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Invertibility preserving mappings onto finite C^*-algebras

by

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Abstract. We prove that every surjective unital linear mapping which preserves invertible elements from a Banach algebra onto a C^*-algebra carrying a faithful tracial state is a Jordan homomorphism, thus generalising Aupetit's 1998 result for finite von Neumann algebras.

1. Introduction. A linear mapping T between two unital, complex Banach algebras is said to be *spectrum-preserving* if, for every element a in the domain algebra, its spectrum $\sigma(a)$ coincides with $\sigma(Ta)$. Provided the codomain is semisimple and T is surjective, T must be bounded (a result belonging to Aupetit [1, Theorem 5.5.2]). Provided the domain is semisimple too, T is injective; this follows from Zemánek's characterisation of the radical [1, Theorem 5.3.1], as for each a such that $Ta = 0$ and for every x,

$$\sigma(a+x) = \sigma(Ta+Tx) = \sigma(Tx) = \sigma(x),$$

which implies that a belongs to the radical, which is zero in the semisimple case. Moreover, $T1 = 1$, that is, T is *unital.* As a result, a surjective spectrum-preserving mapping between semisimple Banach algebras is a topological isomorphism, and one naturally wonders if it is also an isomorphism of (some of) the algebraic structure.

A *Jordan homomorphism* is a linear mapping T with the property $T(a^2) = (Ta)^2$ for all a in the domain (which is equivalent to $T(ab+ba) = TaTb + TbTa$ for all a and b). A Jordan isomorphism turns out to be spectrum-preserving, and a lot of work has been invested to explore to what extent the reverse implication holds. A pleasant survey on the history of this topic is contained in [2]; see also [8, 13, 14] for related questions.

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In [3], Aupetit proved that every surjective spectrum-preserving linear mapping between von Neumann algebras is a Jordan isomorphism. It is not difficult to see that it suffices that one of the algebras is a unital C^*-algebra of real rank zero and the other a unital semisimple Banach algebra. However, the problem remains open for general C^*-algebras. It is also known that the assumption on T can be relaxed to a surjective unital *invertibility-preserving* linear mapping (that is, $\sigma(Ta) \subseteq \sigma(a)$ for all a); the conclusion is then that T is a Jordan homomorphism.

In an earlier paper [2], Aupetit had already obtained the same result for finite von Neumann algebras. The main tool in that result was the Fuglede–Kadison determinant Δ; see [9, p. 105] for its definition and properties. Its relation to the finite trace τ is given by $\Delta(a) = \exp(\tau(\log|a|))$, for every invertible element a. In our approach we bypass the determinant and work exclusively with a (faithful) tracial state instead in order to obtain the following generalisation.

Theorem. *Let B be a unital complex Banach algebra and let A be a unital finite C^*-algebra. Let $T\colon B \to A$ be a surjective unital linear mapping which preserves invertible elements. Then T is a Jordan homomorphism.*

We largely follow Aupetit's arguments, but to emphasise the differences we split up the proof into a series of lemmas in the next section.

2. Preliminaries. Let A be a unital C^*-algebra. We say that A is *finite* if it comes equipped with a *faithful tracial state*, that is, a linear functional τ such that $\tau(1) = 1 = \|1\|$, $\tau(ab) = \tau(ba)$ for all $a, b \in A$ and $\tau(a^*a) = 0$ implies $a = 0$. Such a functional is necessarily positive and bounded.

We denote the set of all *states* of A (positive linear functionals of norm 1) by S and by Sp the subset of all *spectral states* f of A, that is, $f \in S$ and $|f(x)| \leq \rho(x)$ for every $x \in A$, where $\rho(x)$ denotes the spectral radius of x. It is known [7, Theorem 4 in §13] that every $f \in Sp$ has the trace property, that is, $f(ab) = f(ba)$ for all $a, b \in A$, and that $f(a) \in \operatorname{co}\sigma(a)$, the convex hull of the spectrum $\sigma(a)$ of a, for each $a \in A$; see [7, Lemma 2 in §13] or [1, Lemma 4.1.15].

Conversely, every tracial state τ belongs to Sp as follows from the subsequent argument. For $a \in A$, denote by $V(a) = \{f(a) \mid f \in S\}$ its (*algebra*) *numerical range* [7]. As is shown in [5, Lemma], and attributed to [11, §2], $\operatorname{co}\sigma(a) = \bigcap_{b\in G(A)} V(bab^{-1})$, where $G(A)$ stands for the group of invertible elements in A. Clearly, $\tau(a)$ belongs to the right hand side of the above identity, and hence $\tau \in Sp$. (See also [12].)

Lemma 2.1. *Let A be a unital C^*-algebra with faithful tracial state τ. Suppose that $g\colon \mathbb{C} \to A$ is an entire function with values in $G(A)$. Then the mapping $g_\tau\colon \mathbb{C} \to \mathbb{R}$, $g_\tau(\lambda) = \tau(\log(|g(\lambda)|)$, is harmonic.*

Proof. The argument in the proof of Théorème 1.11 in [2], which is already entirely formulated in terms of the trace, takes over verbatim. ■

In the following, T will denote a surjective unital linear mapping defined on a (complex, unital) Banach algebra B with values in a finite unital C^*-algebra A. We will assume that T *preserves invertible elements* so that $TG(B) \subseteq G(A)$. It follows from [1, Theorem 5.5.2] that T is bounded.

Fix $a, b \in B$ and define

$$g\colon \mathbb{C} \times \mathbb{C} \to G(A), \qquad g(\lambda, \mu) = T(e^{\lambda a} e^{\mu b}) e^{-\lambda Ta} e^{-\mu Tb}.$$

Then g is a separately entire function. Its series expansion reads as follows:

$$\begin{aligned} g(\lambda,\mu) = 1 &+ \frac{\lambda^2}{2}\bigl(T(a^2) - (Ta)^2\bigr) + \frac{\mu^2}{2}\bigl(T(b^2) - (Tb)^2\bigr) \\ &+ \lambda\mu\bigl(T(ab) - TbTa\bigr) + \frac{\lambda^3}{6}\bigl(T(a^3) + 2(Ta)^3 - 3T(a^2)Ta\bigr) \\ &+ \frac{\lambda^2\mu}{2}\bigl(T(a^2b) + (Ta)^2Tb + Tb(Ta)^2 - T(a^2)Tb - 2T(ab)Ta\bigr) \\ &+ \frac{\lambda\mu^2}{2}\bigl(T(ab^2) + 2TbTaTb - 2T(ab)Tb - T(b^2)Ta\bigr) \\ &+ \frac{\mu^3}{6}\bigl(T(b^3) + 2(Tb)^3 - 3T(b^2)Tb\bigr) + \text{remainder} \end{aligned}$$

where the remainder only contains terms of degree 4 or higher in λ and μ; we will put it to good use in the proof of the main theorem.

By Lemma 2.1, the function

$$g_\tau\colon \mathbb{C} \times \mathbb{C} \to \mathbb{R}, \qquad g_\tau(\lambda,\mu) = \tau(\log(|g(\lambda,\mu)|)),$$

is separately harmonic in λ and μ and thus there exists a separately entire function $h(\lambda,\mu)$ such that $\operatorname{Re} h(\lambda,\mu) = g_\tau(\lambda,\mu)$ for all $\lambda,\mu \in \mathbb{C}$.

The next step will be to establish the following three lemmas; for their proofs, see Section 3.

Lemma 2.2. *For all $\lambda,\mu \in \mathbb{C}$, we have $e^{g_\tau(\lambda,\mu)} \leq \|g(\lambda,\mu)\|$.*

Lemma 2.3. *With the above notation and caveats, let $g^*(\lambda,\mu)$ stand for $(g(\lambda,\mu))^*$. There exists $r > 0$ such that, for all $\lambda,\mu \in \mathbb{C}$ with $|\lambda|, |\mu| < r$, we have*

$$2\operatorname{Re} h(\lambda,\mu) = \tau\bigl(\log(g^*(\lambda,\mu)g(\lambda,\mu))\bigr) = -\sum_{k=1}^{\infty} \frac{1}{k}\tau\bigl((1 - g^*(\lambda,\mu)g(\lambda,\mu))^k\bigr).$$

Lemma 2.4. *For all λ,μ in a neighbourhood of zero,*

$$\tau\bigl(\log(g^*(\lambda,\mu)g(\lambda,\mu))\bigr) = 0.$$

3. Proofs of the lemmas and the main theorem. The argument of the first lemma differs from [2] in that we cannot make use of the determinant in order to locate the appropriate values in the convex hull of the spectrum.

Proof of Lemma 2.2. As we observed above,

$$\tau(\log(|g(\lambda,\mu)|)) \in \operatorname{co}\sigma(\log(|g(\lambda,\mu)|))$$

for all $\lambda,\mu \in \mathbb{C}$. Since the spectrum of $\log(|g(\lambda,\mu)|)$ is contained in $\mathbb{R}$, it follows that $\operatorname{co}\sigma(\log(|g(\lambda,\mu)|)) = [s,t]$ for some $s,t \in \sigma(\log(|g(\lambda,\mu)|))$ with $s \le t$. Since the exponential function is strictly increasing, the Spectral Mapping Theorem implies that

$$e^{g_\tau(\lambda,\mu)} \in [e^s, e^t] = \operatorname{co}\sigma(e^{\log(|g(\lambda,\mu)|)}) = \operatorname{co}\sigma(|g(\lambda,\mu)|).$$

As a result, $0 < e^{g_\tau(\lambda,\mu)} \le \rho(|g(\lambda,\mu)|)$, and therefore

$$\begin{aligned} e^{2g_\tau(\lambda,\mu)} &\le \rho(|g(\lambda,\mu)|)^2 = \rho(|g(\lambda,\mu)|^2) = \rho(g^*(\lambda,\mu)g(\lambda,\mu)) \\ &= \|g^*(\lambda,\mu)g(\lambda,\mu)\| = \|g(\lambda,\mu)\|^2 \end{aligned}$$

as claimed. ∎

The next argument is rather straightforward.

Proof of Lemma 2.3. As $g(0,0) = T1 = 1$, by continuity, there is $r > 0$ such that, for all λ,μ with $|\lambda|,|\mu| < r$, we have $\|1 - g^*(\lambda,\mu)g(\lambda,\mu)\| < 1$. The series expansion of the logarithm thus yields

$$-\log(g^*(\lambda,\mu)g(\lambda,\mu)) = \sum_{k=1}^{\infty} \frac{1}{k}(1 - g^*(\lambda,\mu)g(\lambda,\mu))^k. \tag{3.1}$$

The definition of g_τ entails that

$$\begin{aligned} 2\operatorname{Re} h(\lambda,\mu) &= 2\tau(\log(|g(\lambda,\mu)|)) = \tau(\log(|g(\lambda,\mu)|^2)) \\ &= \tau\big(\log(g^*(\lambda,\mu)g(\lambda,\mu))\big). \end{aligned}$$

Combining these two identities gives the claim. ∎

The proof of the third lemma follows exactly Aupetit's arguments. (There appears to be some misprint at the bottom of p. 61 and top of p. 62 of [2].)

Proof of Lemma 2.4. For all $\lambda,\mu \in \mathbb{C}$, we have

$$|e^{h(\lambda,\mu)}| = e^{\operatorname{Re} h(\lambda,\mu)} = e^{g_\tau(\lambda,\mu)} \le \|g(\lambda,\mu)\|$$

by Lemma 2.2. Since

$$\|g(\lambda,\mu)\| \le \|T\|\, e^{|\lambda|(\|a\|+\|Ta\|)+|\mu|(\|b\|+\|Tb\|)}$$

it follows that $e^{h(\lambda,\mu)} = e^{\alpha\lambda+\beta\mu+\gamma}$ for suitable $\alpha,\beta,\gamma \in \mathbb{C}$ [4, Lemma 3.2]. As $g_\tau(0,0) = 0$ we have $|e^\gamma| = 1$, thus we may assume that $\gamma = 0$ (since we need only the real part of γ). Therefore, $2\operatorname{Re} h(\lambda,\mu) = \alpha\lambda + \beta\mu + \bar\alpha\bar\lambda + \bar\beta\bar\mu$.

From Lemma 2.3 we obtain

$$\alpha\lambda + \beta\mu + \bar{\alpha}\bar{\lambda} + \bar{\beta}\bar{\mu} = -\sum_{k=1}^{\infty} \frac{1}{k}\tau\big((1 - g^*(\lambda,\mu)g(\lambda,\mu))^k\big)$$

for all λ, μ such that $|\lambda|, |\mu| < r$ for suitable $r > 0$. The series expansion of $g(\lambda,\mu)$ above does not contain any powers of λ or μ of first order, hence the series expansion in (3.1) cannot either. This entails that both α and β are equal to zero.

It now follows from Lemma 2.3 that $\tau(\log(g^*(\lambda,\mu)g(\lambda,\mu))) = 0$. ∎

We now have all the tools to prove our main theorem by adapting the arguments in [2, Theorem 1.12] to our situation.

Proof of the Theorem. Set $f(\lambda,\mu) = \sum_{k=1}^{\infty} \frac{1}{k}\tau((1 - g^*(\lambda,\mu)g(\lambda,\mu))^k)$ for all λ, μ such that $|\lambda|, |\mu| < r$ for suitable $r > 0$ (given by Lemma 2.3). By Lemma 2.4, $f = 0$ and thus

$$\frac{\partial^2}{\partial\lambda\partial\mu} f(0,0) = 0 = \frac{\partial^3}{\partial\lambda^2\partial\mu} f(0,0).$$

Using these identities after substituting in the series expansion of g into the log-series, we find that

$$\tau(T(ab) - TaTb) = 0 \tag{3.2}$$

and

$$\tau\big(T(a^2b) + (Ta)^2Tb + Tb(Ta)^2 - T(a^2)Tb - 2T(ab)Ta\big) = 0 \tag{3.3}$$

for all $a, b \in B$. From (3.2) we obtain

$$\tau(T(a^2b)) = \tau(T(a^2)Tb)$$

and

$$\tau(T(a^2b)) = \tau(T(a(ab))) = \tau(TaT(ab))$$

so that (3.3) reduces to

$$\tau((Ta)^2Tb) = \tau(TaT(ab)),$$

using the trace property. It follows that

$$\tau((Ta)^2Tb) = \tau(T(a^2)Tb).$$

Since T is surjective we may choose $b \in B$ such that $Tb = ((Ta)^2 - T(a^2))^*$ wherefore the last identity yields, for each $a \in B$,

$$\tau\big(((Ta)^2 - T(a^2))((Ta)^2 - T(a^2))^*\big) = 0.$$

The faithfulness of τ implies that T is a Jordan homomorphism. ∎

4. Conclusions. In this section, we collect together some consequences and sharpening of our main theorem. We also relate it to open problems of a similar nature.

Suppose T is a surjective linear mapping between two semisimple unital Banach algebras which preserves the spectrum of each element. Then T is injective (as explained in the Introduction) and $T1 = 1$. The latter follows, for example, from

$$\sigma((T1-1)+Tx) = \sigma(T1+Tx) - 1 = \sigma(1+x) - 1 = \sigma(x) = \sigma(Tx)$$

and the surjectivity of T which entails that $\sigma((T1-1)+y) = \sigma(y)$ for all y in the codomain. Thus, by Zemánek's characterisation of the radical, $T1 - 1 = 0$.

As a result, we have a symmetric situation and can apply the Theorem to either T or its inverse to obtain the following consequence.

Corollary 4.1. *Let T be a surjective spectrum-preserving linear mapping between two semisimple unital Banach algebras. If either of them is a unital C^*-algebra equipped with a faithful tracial state, then T is a Jordan isomorphism.*

This is another contribution to a longstanding, still open problem by Kaplansky, who asked in 1970 whether the above statement holds without any further assumptions on the Banach algebras. For further references, see [2, 10].

All the steps in the proof of the Theorem but the very last one can be performed for each individual tracial state on a unital C^*-algebra. Therefore, the assumption can be relaxed to the existence of a faithful family of tracial states, that is, a family $\{\tau_i \mid i \in I\}$ of tracial states τ_i such that $\tau_i(a^*a) = 0$ for all $i \in I$ implies $a = 0$.

In particular, since any tracial state on a simple unital C^*-algebra is faithful, we obtain the following result.

Corollary 4.2. *Let $T\colon B \to A$ be a surjective unital invertibility-preserving linear mapping into a simple unital C^*-algebra A which carries a tracial state. Then T is a Jordan homomorphism.*

Remark 4.3. Our terminology of a "finite" C^*-algebra is not quite standard. In [6, III.1.3.1], a unital C^*-algebra A is called *finite* if the identity of A is a finite projection; that is, there is no proper subprojection which is Murray–von Neumann equivalent to 1. Every unital C^*-algebra with a faithful tracial state is finite in this sense, but the converse fails in general (though it holds for stably finite exact C^*-algebras). We prefer here a definition that does not make reference to any projections.

We can also strengthen our main theorem in a different direction. Let $G_1(B)$ denote the *principal component of* $G(B)$, where B is a unital Banach algebra. It is known (see, e.g., [1, Theorem 3.3.7]) that

$$G_1(B) = \{e^{x_1} \cdots e^{x_n} \mid x_i \in B,\, n \in \mathbb{N}\}.$$

The associated *exponential spectrum* of $x \in B$ is

$$\sigma_\varepsilon(x) = \{\lambda \in \mathbb{C} \mid \lambda - x \notin G_1(B)\}.$$

In certain situations it is more natural and expedient to consider the exponential spectrum instead of the smaller spectrum; see, e.g., [1, Theorem 3.3.8]. From the proof of our main result we see that it suffices that the mapping T sends the product of any two exponentials in B onto an invertible element in A. This gives the following corollary.

COROLLARY 4.4. *Let B be a unital complex Banach algebra and let A be a unital finite C^*-algebra. Let $T\colon B \to A$ be a surjective unital linear mapping such that $TG_1(B) \subseteq G(A)$. Then T is a Jordan homomorphism.*

A *spectral isometry* between two Banach algebras A and B is a linear mapping S such that $\rho(Sx) = \rho(x)$ for all $x \in A$. Clearly, every spectrum-preserving mapping is a spectral isometry and so is every Jordan isomorphism. A conjecture related to Kaplansky's problem mentioned above states that every unital surjective spectral isometry between two C^*-algebras is a Jordan isomorphism. This conjecture has been confirmed in many cases (see, e.g., [13, 14]), but is open in all generality. Notably, it was verified in [15] if A is a unital C^*-algebra of real rank zero and without tracial states. The above Corollary 4.1 is thus a step forward in the direction of confirming the general conjecture.

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