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Towards a non-selfadjoint version of Kadison's theorem

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Abstract

Kadison's theorem of 1951 describes the unital surjective isometries between unital C^*-algebras as the Jordan *-isomorphisms. We propose a non-selfadjoint version of his theorem and discuss the cases in which this is known to be true.

Key Words: Spectral isometry, Jordan isomorphism, C^*-algebra.

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1. Introduction

Among the most important linear mappings between Banach spaces are the isometries; no wonder therefore that they have been given a lot of attention. One of the best-known results is the classical Banach–Stone theorem, proved by Banach in 1932 under the assumption of separability and by Stone in 1937 in the general case.

Theorem 1.1. (Banach–Stone) *Let X, Y be compact Hausdorff spaces. Let $T\colon C(X) \to C(Y)$ be a surjective linear isometry between the associated Banach spaces of complex-valued continuous functions. Then there exist a uniquely determined function $h \in C(Y)$ with $|h| = 1$ and a uniquely determined homeomorphism $\varphi\colon Y \to X$ such that $Tf = h\,(f \circ \varphi)$ for all $f \in C(X)$.*

In particular, if T is *unital*, that is, $T1 = 1$, then T is multiplicative, hence an algebra isomorphism. Note that we get for free that T preserves the canonical involution on the spaces of continuous functions: $T\bar{f} = \overline{Tf}$, where $\bar{f}$ denotes the complex-conjugate function.

In 1951 Kadison obtained the following generalisation of the above result to arbitrary C^*-algebras [8].

Theorem 1.2. (Kadison) *Let A, B be unital C^*-algebras. Let $T\colon A \to B$ be a surjective linear isometry between A and B. Then there exist a uniquely determined unitary $u \in B$ and a uniquely determined Jordan *-isomorphism $\Phi\colon A \to B$ such that $Tx = u\,\Phi x$ for all $x \in A$.*

Here, a Jordan *-isomorphism Φ is a bijective linear mapping with the property that $\Phi(x^2) = (\Phi x)^2$ for all $x \in A$ and which is *selfadjoint*, i.e. preserves selfadjoint elements. It follows easily that Φ indeed preserves the Jordan product $x \cdot y = \frac{1}{2}(xy+yx)$, $x, y \in A$. Sometimes such mappings are referred to as *C^*-isomorphisms* and in a certain sense they can be built from isomorphisms and anti-isomorphisms, see [1, Section 6.3] for example.

Every Jordan *-isomorphism Φ is an isometry. Indeed, let $x \in A$ be positive; then $x = y^2$ for some $y \in A_{sa}$. Hence, $\Phi x = \Phi(y^2) = (\Phi y)^2$ is positive. By the Russo–Dye theorem [15, Corollary 2.9], $\|\Phi\| = \|\Phi 1\|$ and since it is easily seen that every Jordan *-isomorphism is unital, it follows that Φ is a contraction. Applying the same argument to Φ^{-1} yields the claim.

Part of Kadison's argument establishes the fact that $u = \Phi 1$ is a unitary in B, whenever Φ is a surjective isometry. Therefore one can reduce to the case of a *unital* isometry and thus Kadison's theorem is in fact a characterisation of the unital surjective isometries between unital C^*-algebras as the Jordan *-isomorphisms. A similar reduction applies to the more general mappings discussed below, hence we will from this point on deal exclusively with unital mappings. On the other hand, it is well known that the assumption of surjectivity is inevitable.

Suppose that $T\colon A \to B$ is a unital surjective isometry; then T is selfadjoint. Indeed, let $a \in A_{sa}$, $\|a\| = 1$ and write $Ta = b + ic$, where b, $c \in B_{sa}$. If $c \neq 0$ then there is $\gamma \in \sigma(c)$, the spectrum of c, which is non-zero; we can assume that $\gamma > 0$. Since T is an isometry, for large $n \in \mathbb{N}$, we find

$$\|a + in\|^2 = 1 + n^2 < (\gamma + n)^2 \leq \|c + n\|^2 \leq \|T(a + in)\|^2.$$

This entails that $c = 0$ and so $Ta \in B_{sa}$.

We thus observe that unital isometries are *intrinsically* selfadjoint. In the sequel we wish to discuss a concept of 'non-selfadjoint' isometries that is capable of characterising not necessarily selfadjoint Jordan isomorphisms in a way analogous to Kadison's theorem.

2. Spectral isometries

Suppose that $T\colon A \to B$ is a Jordan isomorphism between the unital C^*-algebras A and B. It is well known that $a \in A$ is invertible if and only if $Ta \in B$ is invertible; see, e.g., [7, Lemma 4.1]. Consequently T preserves the spectrum of every element, that is, $\sigma(Ta) = \sigma(a)$ for every $a \in A$. A fortiori T preserves the spectral radius $r(a)$, and it turns out that this is the decisive property.

Definition 2.1. A linear mapping $T\colon A \to B$ between two unital C^*-algebras is called a *spectral isometry* if $r(Ta) = r(a)$ for every $a \in A$.

Since norm and spectral radius coincide in the commutative case, it is evident that we only obtain a new notion if at least A or B is not commutative. However, it turns out that in fact *both* have to be non-commutative, otherwise we are back to the notion of an isometry.

Proposition 2.2. *Let $T\colon A \to B$ be a unital surjective spectral isometry between the unital C*-algebras A and B. If A or B is commutative then T is a multiplicative isomorphism.*

This result follows easily from the results in [10] and the Banach–Stone theorem. More generally every surjective spectral isometry restricts to an isomorphism of the centres of general C^*-algebras. To see this, let us first note two properties.

(1) Every spectral isometry is injective.

(2) Every surjective spectral isometry preserves central elements.

Suppose $T\colon A \to B$ is a spectral isometry, and let $a \in A$ be such that $Ta = 0$. For $x \in A$ we obtain $r(a + x) = r(Ta + Tx) = r(Tx) = r(x)$; hence, by Zemánek's characterisation of the radical [2, Theorem 5.3.1], a belongs to the radical of A which is zero. Thus $a = 0$ and (1) holds.

Now assume in addition that T is surjective. Let $z \in Z(A)$, the centre of A. For $b \in B$ take $a \in A$ such that $Ta = b$. Then

$$r(Tz + b) = r(T(z + a)) = r(z + a) \leq r(z) + r(a) = r(Tz) + r(b).$$

By Pták's characterisation of the centre [16, Proposition 2.1] it follows that $Tz \in Z(B)$. This shows (2).

Combining these properties with the Banach–Stone theorem and applying (2) to the spectral isometry $T^{-1}\colon B \to A$, we obtain the stated result.

Proposition 2.3. *Let $T\colon A \to B$ be a unital surjective spectral isometry between the unital C*-algebras A and B. Then $T_{|Z(A)}$ induces a *-isomorphism between $Z(A)$ and $Z(B)$.*

We shall soon make good use of this result. But let us first compare the two concepts of isometry and spectral isometry more closely. Every unital surjective isometry between unital C^*-algebras is a Jordan *-isomorphism by Theorem 1.2, hence a spectral isometry. We remark in passing that we do not know of a direct argument proving this without using Kadison's theorem. Conversely, every selfadjoint unital surjective spectral isometry is an isometry. To see this, let $a \in A_+$, $\|a\| = 1$. Then $\|a-1\| \leq 1$ and therefore $\|Ta\| = 1$ and $\|Ta-1\| \leq 1$, since $Ta \in B_{sa}$ and norm and spectral radius coincide for selfadjoint elements. Consequently Ta is positive which shows that T is a positive map. Applying the Russo–Dye theorem once again we deduce that $\|T\| = \|T1\| = 1$ so T is a contraction. The same argument for T^{-1} yields the result.

Proposition 2.4. *Let $T: A \to B$ be a unital surjective linear map. Then T is an isometry if and only if T is a selfadjoint spectral isometry.*

A 35-year old problem of Kaplansky [9] asks whether every surjective spectrum-preserving linear mapping between unital C^*-algebras has to be a Jordan isomorphism. An important step forward was made by Aupetit [4] by establishing the result for von Neumann algebras. However, to-date no answer appears to be known if neither of the C^*-algebras is real rank zero. Nevertheless, Kaplansky's question together with the above evidence made us surmise the following in [12].

Conjecture 2.5. *Every unital surjective spectral isometry between unital C^*-algebras is a Jordan isomorphism.*

Evidently this conjecture is harder than Kaplansky's; the point we wish to make here is that the statement provides a non-selfadjoint analogue of Kadison's theorem.

In the remainder of this note we shall explain what by now is known on Conjecture 2.5 and discuss some of the techniques involved in proving our results.

3. The theorem

Before stating the main result and discussing the ingredients of its proof we need two more properties of spectral isometries.

(3) Every surjective spectral isometry is bounded (and hence open).

(4) Every surjective spectral isometry preserves nilpotent elements.

Both properties in fact hold for the wider class of spectrally bounded operators. A linear mapping $T: A \to B$ is said to be *spectrally bounded* if there is a constant $M > 0$ such that $r(Tx) \leq M\, r(x)$ for all $x \in A$. The surjectivity and the *semisimplicity of* B then yield the boundedness of T; see [2, Theorem 5.5.1] and, slightly more general, [5]. This gives (3). Property (4) was obtained in [13, Lemma 3.1], once again for surjective spectrally bounded maps. It follows that, if T is a surjective spectral isometry and $a \in A$, then $a^n = 0$ if and only if $(Ta)^n = 0$. Spectrally bounded maps originally were introduced in connection with the non-commutative Singer–Wermer conjecture, see [6] for more details. A number of their basic properties are discussed in [12].

Apart from the commutative situation, which is somewhat special, an important technique employed by many authors to show that a spectral isometry (or, more generally, a spectrally bounded operator) has the Jordan property has been to evaluate it on projections. This, of course, only works if the domain is well supplied with projections. In fact, we do not know of any result that goes beyond the scope of C^*-algebras with real rank zero at present. Indeed, Aupetit's theorem [4] does not rely on the structure of von Neumann algebras but extends to C^*-algebras of real rank zero, see, e.g., [11, Theorem 1.1].

The reason for this approach is the following result, by now standard and being used by many authors.

Proposition 3.1. *Let $T\colon A \to B$ be a bounded linear operator between the C*-algebras A and B. Suppose that A has real rank zero. If T maps projections in A onto idempotents in B then T is a Jordan homomorphism, that is, $T(x^2) = (Tx)^2$ for all $x \in A$.*

The idea of the argument is as follows. If $p \in A$ is a projection then, by assumption, $Tp \in B$ is an idempotent. If $q \in A$ is a projection orthogonal to p, then an easy argument shows that the idempotent Tq is orthogonal to Tp. Hence, if $a \in A$ is of the form $a = \sum_{j=1}^{n} \lambda_j p_j$ for some scalars λ_j and finitely many mutually orthogonal projections p_j, then

$$T(a^2) = T\Big(\sum_{j=1}^{n} \lambda_j^2 p_j\Big) = \sum_{j=1}^{n} \lambda_j^2 T p_j = (Ta)^2.$$

The assumption on A to have real rank zero amounts to the fact that every self-adjoint element can be approximated by elements of the above form; hence the continuity of T entails the Jordan property on A_{sa}. Finally, the cartesian decomposition $x = a + ib$, $a, b \in A_{sa}$ completes the argument. For more details see [13, Lemma 2.1].

Combining Proposition 3.1 with property (3) above opens up the way to deal with spectral isometries.

Corollary 3.2. *Let $T\colon A \to B$ be a unital surjective spectral isometry between the unital C*-algebras A and B. If A has real rank zero and T maps projections in A onto idempotents in B then T is a Jordan isomorphism.*

We now state the result which, to our knowledge, is the most general so far.

Theorem 3.3. *Let $T\colon A \to B$ be a unital surjective spectral isometry between the unital C*-algebras A and B. If either*

(i) *A is a von Neumann algebra without direct summand of type* II_1

or

(ii) *A is a simple C*-algebra with real rank zero and without tracial states*

then T is a Jordan isomorphism.

Outline of proof. In view of Corollary 3.2 our aim is to show that, whenever $p \in A$ is a projection, then Tp is an idempotent. Let $q = 1 - p$ and suppose, without loss of generality, that $p \neq 0 \neq q$. If A satisfies the assumptions in (ii), then every element in the subalgebras pAp and qAq is a finite sum of elements of square zero. This follows from results by Marcoux, Pop and Zhang, see [11]. If A is a properly infinite von Neumann algebra, then we can reduce to the case where both pAp and qAq are properly infinite and, by using results due to Pearcy and Topping, obtain the same statement; for details see [13].

Hence, there are finitely many $a_i \in pAp$, $b_j \in qAq$ such that $p = \sum_i a_i$, $q = \sum_j b_j$, and $a_i^2 = b_j^2 = 0$ for all i, j. We claim that

$$(Tp)(Tq) + (Tq)(Tp) = 0 \tag{3.1}$$

which implies that

$$2\,(Tp - (Tp)^2) = (Tp)(1 - Tp) + (1 - Tp)(Tp) = 0,$$

as $T1 = 1$. Consequently, Tp is idempotent.

Since $(a_i + b_j)^2 = 0$ for all i, j, property (4) above entails that $(T(a_i + b_j))^2 = 0$ for all i, j. On the other hand,

$$(T(a_i+b_j))^2 = (Ta_i)^2+(Ta_i)(Tb_j)+(Tb_j)(Ta_i)+(Tb_j)^2 = (Ta_i)(Tb_j)+(Tb_j)(Ta_i),$$

wherefore $(Ta_i)(Tb_j) + (Tb_j)(Ta_i) = 0$ for all i, j. Summing over all indices yields the claim (3.1).

If A is a general von Neumann algebra we write it in its type decomposition, but under the hypothesis (i), we can assume that the type II_1 part is absent:

$$A = A_{\mathrm{I}_{fin}} \oplus A_{\mathrm{I}_\infty} \oplus A_{\mathrm{II}_\infty} \oplus A_{\mathrm{III}}.$$

Now comes an important step. Each of the direct summands above is of the form eA for some central projection e in A. By Proposition 2.4 we know that $f = Te$ is a central projection in B. But, in addition, $T(ex) = (Te)(Tx)$ for all $x \in A$ and so T maps the C^*-subalgebra eA onto the C^*-subalgebra fB. It follows that T restricts to a unital surjective spectral isometry from eA onto fB. This is obtained in [14]. As a result, we can treat each of the parts separately, since T will be a Jordan isomorphism if and only if each of the restrictions is.

The last three summands we already dealt with as they are properly infinite; so it remains to cover the finite type I case. In other words, we can assume that A is of the form $A = \prod_{n\in\mathbb{N}} C(X_n, M_n)$, where each X_n is a hyperstonean space and M_n denotes the complex $n \times n$ matrices. Since each of the von Neumann subalgebras $C(X_n, M_n)$ once again is of the form eA for a central projection $e \in A$, we can employ the same reduction argument as above in order to assume that, in fact, $A = C(X, M_n)$ for some hyperstonean space X and some $n \in \mathbb{N}$.

Since the centre $Z(A)$ is isomorphic to $C(X)$ and is generated by its projections, an argument as in Proposition 3.1 gives us the identity $T(zx) = (Tz)(Tx)$ for all $z \in Z(A)$, $x \in A$ from the analogous identity for central projections e noted above. Let I be a Glimm ideal of A, that is, an ideal of the form $I = MA$ for a (unique) maximal ideal M of $Z(A)$. It follows that $J = TI$ is a Glimm ideal of B, since $TI = T(MA) = NB$, where $N = TM$ is a maximal ideal in $Z(B)$ by Proposition 2.3. Every Glimm ideal of A is in fact a maximal ideal, as it is of the form

$$I = \{f \in C(X, M_n) \mid f(t) = 0 \text{ for some } t \in X\},$$

and the quotient A/I is isomorphic to M_n. The induced unital mapping $\hat{T}\colon A/I \to B/J$ turns out to be a spectral isometry onto B/J, by [14, Proposition 9]. Since $\dim A/I = n^2$ and T is a linear isomorphism, B/J is a finite-dimensional C^*-algebra of dimension n^2 with trivial centre (which is isomorphic to $Z(A/I) = \mathbb{C}$). Consequently, $\hat{T}$ in fact is a unital surjective spectral isometry from M_n to M_n. Each such spectral isometry has been shown to be a Jordan isomorphism in [3, Proposition 2]. Since the Glimm ideals separate the points, it finally follows that T is a Jordan isomorphism, and the proof is complete. □

Slight extensions beyond the situation of C^*-algebras covered by condition (ii) in Theorem 3.3 are possible, but do not give insight into the open unknown cases. These are, on the one hand, C^*-algebras not of real rank zero; here, even the case $C([0,1], M_n)$ appears to be open at the time of this writing, and on the other hand, finite von Neumann algebras; e.g., the case of the hyperfinite II_1 factor is still unsettled. It is intriguing that the non-selfadjoint version of Kadison's theorem needs, at least at present, different techniques for different types of algebras whereas the characterisation of onto isometries allows for such an elegant and comprehensive proof.

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