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CHARACTERIZING JORDAN HOMOMORPHISMS

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ABSTRACT. It is shown that every bounded, unital linear mapping that preserves elements of square zero from a C^*-algebra of real rank zero and without tracial states into a Banach algebra is a Jordan homomorphism.

Throughout, A and B will be (at least) unital complex Banach algebras. A *Jordan homomorphism* between A and B is a linear mapping $T\colon A \to B$ such that $T(x^2) = (Tx)^2$ for all $x \in A$. That is, T is a homomorphism of the associated special Jordan algebras A^+ and B^+. The quest of characterizing Jordan homomorphisms by spectral conditions is largely motivated by Kaplansky's question from the 1970s whether every unital, bounded, surjective linear mapping T which preserves invertibility between semisimple Banach algebras must be a Jordan epimorphism (i.e., a surjective Jordan homomorphism). All listed conditions are well known to be necessary (where *unital* means $T1 = 1$). While this problem is open in the stated generality, many contributions under additional conditions on the algebras have been obtained (see, e.g., the survey [9]), notably if the domain is a C^*-algebra. Aupetit [3] affirmed Kaplansky's question for von Neumann algebras, and his proof can easily be adapted to the situation where A is a unital C^*-algebra with real rank zero and B is any semisimple unital Banach algebra; cf. [8, Theorem 1.1].

The abundance of projections in C^*-algebras with real rank zero (i.e., every self-adjoint element can be approximated by finite linear combinations of orthogonal projections) is a key ingredient via the following result; see [10, Lemma 2.1] or [7, Lemma 1].

Lemma 1. *Let $T\colon A \to B$ be a bounded linear operator from a C^*-algebra A with real rank zero into a Banach algebra B sending projections in A to idempotents in B. Then T is a Jordan homomorphism.*

With few exceptions, see, e.g., [11], the assumption of real rank zero in the domain has remained essential. The connection between idempotents and elements of square zero is provided by the following lemma from [10, Lemma 3.3].

Lemma 2. *Let $T\colon A \to B$ be a linear mapping between Banach algebras A and B which preserves elements with square zero. If e, f are orthogonal idempotents in A, then*

$$(Ta)(Tb) + (Tb)(Ta) = 0$$

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for all $a \in eAe$, $b \in fAf$ which can be written as finite sums of elements with square zero.

Let A be a unital C^*-algebra and denote by $N^{(2)}$ the linear span of all elements in A with square zero and by $[A,A]$ the linear span of all commutators $[x,y]=xy-yx$, $x,y \in A$. Every square-zero element x is a commutator; e.g., write $x=\big[v|x|^{1/2},|x|^{1/2}\big]$ where $x=v|x|$ is the polar decomposition of x (in a faithful representation of A or the enveloping von Neumann algebra A^{**}). Indeed, $(|x|^{1/2}v|x|^{1/2})(|x|^{1/2}v|x|^{1/2})^* = |x|^{1/2}xv^*|x|^{1/2} = 0$ since $|x|^{1/2}$ belongs to the C^*-subalgebra generated by x^*x and $(x^*x)x=0$. Consequently, $N^{(2)} \subseteq [A,A]$ in general.

Recall that a *tracial state* on A is a positive linear functional τ of norm 1 such that $\tau(xy)=\tau(yx)$ for all $x,y\in A$. Clearly $[A,A]\subseteq \ker\tau$ for every tracial state τ. Therefore, an obstruction for the equality $A=[A,A]$ is the existence of a tracial state. Pop [13] proved that this is the only obstruction. On the other hand, Robert showed in [14, Theorem 4.2] that $[A,A]=N^{(2)}$ provided A has no characters. Combining these two results we obtain a characterization of the identity $N^{(2)}=A$.

Lemma 3. *Let A be a unital C^*-algebra. Then every element of A is a finite sum of square zero elements if and only if A has no tracial states.*

This is clear as every character is a tracial state.

We now put the results above together to verify a conjecture, in the real rank zero case, which we first stated in [8]; see also [6].

Theorem. *Let A be a unital C^*-algebra with real rank zero and $\dim A \geq 2$. The following conditions are equivalent.*

(a) *A has no tracial states;*
(b) *every bounded unital linear mapping from A into a unital Banach algebra preserving square zero elements is a Jordan homomorphism.*

Proof. Evidently we have to exclude the case $\dim A < 2$ since every unital linear mapping is multiplicative in this case.

Suppose τ is a tracial state on A. Since $N^{(2)} \subseteq [A,A] \subseteq \ker\tau$, τ preserves square zero elements. Therefore, assumption (b) entails that τ is a character on A. Hence $A=\ker\tau\oplus\mathbb{C}1$ and $\ker\tau$ is a maximal ideal. Assume first that A is not commutative. Let S be an *-automorphism of A and pick $\gamma\in(0,1)$. Set

$$T\colon A\to A,\ Tx=\gamma\, Sx+(1-\gamma)\,\tau(x).$$

Then T is a unital bounded linear mapping on A, with $\|T\|=1$, and, for all $x,y\in A$,

$$(Tx)(Ty)=\gamma^2\,(Sx)(Sy)+(1-\gamma)^2\,\tau(x)\tau(y)+\gamma(1-\gamma)\,\tau(y)Sx+\gamma(1-\gamma)\,\tau(x)Sy.$$

Take $x\in A$ non-zero such that $x^2=0$ (such x exists only if A is noncommutative) and put $y=x$. Then $(Tx)^2=\gamma^2\,(Sx)^2=\gamma^2\,S(x^2)=0$ so T preserves elements of square zero and hence is a Jordan homomorphism under condition (b). On the other hand, for $y = x^*$,

$$(Tx)(Ty) + (Ty)(Tx) = \gamma^2 \big((Sx)(Sy) + (Sy)(Sx)\big) = \gamma^2\, S(xy + yx)$$
$$\neq \gamma\, S(xy + yx) = T(xy + yx),$$

a contradiction. Assume now that A is commutative. Since $\dim A \geq 2$ there is another character $\tau' \neq \tau$ of A. Take $x, y \in A$ such that $\tau(x) = 0$, $\tau'(x) = 1$, $\tau(y) = 1$ and $\tau'(y) = 0$. Put $\rho = \frac{1}{2}(\tau + \tau')$ which is another tracial state thus a character, by condition (b). Then

$$\rho(xy) = \tfrac{1}{2}(\tau(x)\tau(y) + \tau'(x)\tau'(y)) = 0 \neq \tfrac{1}{4} = \rho(x)\rho(y)$$

which is a contradiction.

Consequently, condition (b) rules out the existence of tracial states on A, that is, condition (a) holds.

Conversely assume condition (a). Let $T\colon A \to B$ be a bounded linear mapping from A into a unital Banach algebra B with $T1 = 1$ and suppose that $(Tx)^2 = 0$ for every $x \in A$ with $x^2 = 0$. Let p be a non-trivial projection in A. As every bounded trace τ on the hereditary C^*-subalgebra pAp extends to A, pAp has no tracial states. (This is known and follows, e.g., from combining [4, Proposition II.4.2] to extend τ from the (completely) full hereditary C^*-algebra to the closed ideal $\overline{ApA}$ and then onto all of A, a slick proof of the latter is contained in [15, Lemma 3.1].) Lemma 3 thus entails that every element in pAp is a finite sum of square zero elements. By Lemma 2, we conclude that

$$0 = (Tp)(T(1-p)) + (T(1-p))(Tp) = 2\,Tp - (Tp)^2$$

and thus Tp is an idempotent in B. Now Lemma 1 finishes the proof. □

The above result extends Theorem 2.4 in [6] for the purely infinite case. Note that every bounded linear mapping which vanishes on elements of square zero necessarily is a trace as $\overline{N^{(2)}} = \overline{[A, A]}$ ([1, Proposition 2.2] or [14, Corollary 2.3]).

A linear mapping $T\colon A \to B$ is said to be *spectrally bounded* if $r(Tx) \leq M\, r(x)$ for some constant $M \geq 0$ and all $x \in A$. Here, $r(\,\cdot\,)$ denotes the spectral radius. When B is semisimple and T is surjective, spectral boundedness implies boundedness of T [2, Theorem 5.5.2]. Every Jordan epimorphism T is unital and preserves invertibility [5, Lemma 4.1]; hence, it is spectrally bounded with constant 1.

Suppose that A is a unital C^*-algebra and B is a unital semisimple Banach algebra. It was observed in [10, Lemma 3.1] that surjective spectrally bounded operators preserve nilpotency, we restrict ourselves to elements of square zero here.

Lemma 4. *Let $T\colon A \to B$ be spectrally bounded and surjective. For every $x \in A$ with $x^2 = 0$ we have $(Tx)^2 = 0$.*

Corollary. *Let $T\colon A \to B$ be a unital surjective spectrally bounded operator from a unital C^*-algebra A with real rank zero and without tracial states onto a unital semisimple Banach algebra B. Then T is a Jordan epimorphism.*

This is now immediate from the theorem and Lemma 4.

The statement of the corollary was obtained under the additional assumption that A is simple in [8, Theorem 3.1]. For properly infinite von Neumann algebras it is in [10, Theorem 3.6] and for A purely infinite of real rank zero in [6, Corollary 2.5] (which extended the simple case treated in [7, Theorem B]).

In the case when the domain C^*-algebra has a tracial state, the above statement fails already in the finite-dimensional situation. In fact, a unital surjective linear mapping $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is spectrally bounded if and only if it is of the form $Tx = \gamma\, Sx + (1-\gamma)\,\tau(x)$, $x \in M_n(\mathbb{C})$ for a unique non-zero complex number γ and a unique Jordan isomorphism S on $M_n(\mathbb{C})$, where τ denotes the unique tracial state on $M_n(\mathbb{C})$; cf. [9], Example 5.3 and Remark 5.5.

On the other hand the assumption that T is unital can sometimes be relaxed, see for example [9], in particular when T is a *spectral isometry*, that is, $r(Tx) = r(x)$ for all $x \in A$; see [12], Proposition 2.3 and Corollary 2.4. The above corollary also removes an additional assumption on the primitive ideal space of the domain algebra from [11, Theorem 3.6].

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