Local Multipliers and Derivations, Sheaves of C-Algebras and Cohomology*

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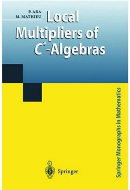
Part I: C^*-algebras of local multipliers

joint work with Pere Ara (Barcelona)



P. ARA AND M. MATHIEU, Local multipliers of C^*-algebras, Springer-Verlag, London, 2003.

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P. Ara and M. Mathieu, *Maximal C^*-algebras of quotients and injective envelopes of C^*-algebras*, Houston J. Math. **34** (2008), 827–872.

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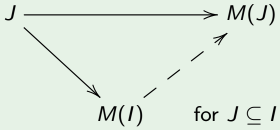
Definition

For every C^*-algebra A,

$$M_{\rm loc}(A) = \varinjlim_{I \in \mathscr{I}_{ce}(A)} M(I),$$

is its *local multiplier algebra*, where



$\mathscr{I}_{ce}(A)$ the filter of all closed essential ideals of A;

$M(I) = \{y \in B(H) \mid yI + Iy \subseteq I\}$ *multiplier algebra of* I.

1978 Pedersen introduces $M_{\mathrm{loc}}(A)$

Theorem

Let A be a separable C-algebra. Every derivation $d\colon A \to A$ extends uniquely to a derivation $d\colon M_{\mathrm{loc}}(A) \to M_{\mathrm{loc}}(A)$ and there is $y \in M_{\mathrm{loc}}(A)$ such that $d = \operatorname{ad} y$ (that is, $dx = [x, y] = xy - yx$ for all $x \in A$).*

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Question

Is $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$ for every C^*-algebra A?

in general, $A \subseteq M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \subseteq \dots$

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positive answer for

- A commutative;
- A simple $(M_{\mathrm{loc}}(A) = M(A))$;
- A AW^*-algebra, in particular von Neumann algebra $(M_{\mathrm{loc}}(A) = A)$.

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common features of last two:

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common features of last two:

- $A \subseteq M_{\text{loc}}(A) \subseteq M_{\text{loc}}(M_{\text{loc}}(A)) \subseteq \ldots \subseteq I(A)$, the injective envelope
- formulas for $M_{\text{loc}}(A)$ and $I(A)$

A commutative:

$$M_{\mathrm{loc}}(A) = \varinjlim_{U \in \mathcal{D}} C_b(U) = \mathrm{alg}\varinjlim_{T \in \mathcal{T}} C_b(T) = I(A),$$

where $\mathcal{D}$ dense open; $\mathcal{T}$ dense G_δ subsets of $\mathrm{Prim}(A)$.

Hence $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(I(A)) = I(A) = M_{\mathrm{loc}}(A)$

since $I(A)$ is an AW^*-algebra.

A non-commutative, e.g., $A = C(X, B(H))$:

$$M_{\mathrm{loc}}(A) = \varinjlim_{U \in \mathcal{D}} C_b(U, B(H)_\beta)$$
$$\subseteq \operatorname{alg} \varinjlim_{T \in \mathcal{T}} C_b(T, B(H)_\beta) = I(A),$$

where $\mathcal{D}$ dense open; $\mathcal{T}$ dense G_δ subsets of Stonean space X.

Depending on properties of X, $\subseteq$ can be strict and still $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = I(A)$!

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1991 Ara–Mathieu obtain *local Dauns–Hofmann theorem*

$$Z(M_{\mathrm{loc}}(A)) = \varinjlim_{I \in \mathscr{I}_{ce}(A)} Z(M(I)) \qquad \text{and hence}$$

$$Z(M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))) = Z(M_{\mathrm{loc}}(A)) \quad \text{for every } A.$$

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2011 Ara–Mathieu provide comprehensive explanation and general procedure to produce examples as well as positive cases

Main Theorem

Theorem

Let B and C be separable C-algebras and suppose that at least one of them is nuclear. Suppose further that B is simple and non-unital and that* $\mathrm{Prim}(C)$ *contains a dense* G_δ *subset consisting of closed points. Let* $A = C \otimes B$*. Then*

$$M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$$

if and only if $\mathrm{Prim}(A)$ *contains a dense subset of isolated points.*

Consequence

Corollary

Let X be a perfect, second countable, locally compact Hausdorff space. Let $A = C_0(X) \otimes B$ for some non-unital separable simple C-algebra B. Then $M_{\mathrm{loc}}(A) \neq M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$.*

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Thus it is easy to answer Pedersen's question in the negative!

Outline of proof

"if" part:

Let $X = \mathrm{Prim}(A)$, X_1 the set of isolated points in X and $X_2 = X \setminus \overline{X_1}$. Then X_1 and X_2 are open subsets of X with corresponding closed ideals $I_1 = A(X_1)$ and $I_2 = A(X_2)$ of A. If X_1 is dense, I_1 is the minimal essential closed ideal of A so $M_{\mathrm{loc}}(A) = M(I_1)$. It follows that

$$M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(M(I_1)) = M_{\mathrm{loc}}(I_1) = M_{\mathrm{loc}}(A).$$

Outline of proof

"only if" part:

In the general case, $M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(I_1) \oplus M_{\mathrm{loc}}(I_2)$.
If $X_2 \neq \emptyset$, it contains a dense G_δ subset of closed points and so $I_2 = C(X_2) \otimes B$ while X_2 is a perfect space. It follows that

$$\begin{aligned} M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) &= M_{\mathrm{loc}}(M_{\mathrm{loc}}(I_1) \oplus M_{\mathrm{loc}}(I_2)) \\ &= M_{\mathrm{loc}}(M_{\mathrm{loc}}(I_1)) \oplus M_{\mathrm{loc}}(M_{\mathrm{loc}}(I_2)) \end{aligned}$$

hence it suffices to show that $M_{\mathrm{loc}}(I_2) \neq M_{\mathrm{loc}}(M_{\mathrm{loc}}(I_2))$ or, in other words, we can assume that X is perfect.

Outline of proof

the main part of the proof of the *"only if"* direction uses a combination of algebraic results on the ideal structure of $M_{\mathrm{loc}}(A)$ and a careful study of the topological properties of $\mathrm{Prim}(A)$ together with the monotone completeness of $I(A)$;

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What happens in the unital case?

A dichotomy answer to Pedersen's question

X perfect compact metric space

B separable simple (nuclear) C^*-algebra

(Elliott's programme)

$\Longrightarrow$

$$A = C(X) \otimes B$$

B unital: $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$

B non-unital: $M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A)$

Outline of proof of Main Theorem

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Definition

K_A is the closure of the set of all elements of the form $\sum_{n\in\mathbb{N}} a_n z_n$, where $\{a_n\} \subseteq A$ is a bounded family and $\{z_n\} \subseteq Z = Z(M_{\mathrm{loc}}(A))$ consists of mutually orthogonal projections.

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Lemma 1

K_A is an essential ideal in $M_{\mathrm{loc}}(A)$.

Outline of proof of Main Theorem

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Lemma 2

If $K_I = K_A$ *for all* $I \in \mathscr{I}_{ce}(A)$ *then* $M_{\mathrm{loc}}(K_A) = M(K_A)$.

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Lemma 2

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Lemma 3

Let $y \in I(A)$. *If* $ya \in K_A$ *for all* $a \in A$ *then* $y \in M(K_A)$.

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Proposition

$M^{(3)}_{\mathrm{loc}}(A) = M^{(2)}_{\mathrm{loc}}(A) = M(K_A)$.

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now, $A = C \otimes B$ as in the Theorem, such that $\mathrm{Prim}(A)$ contains a dense G_δ subset consisting of closed points and $\mathrm{Prim}(A)$ is perfect

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recall: $t \in \mathrm{Prim}(A)$ is *separated* if t and every point $t' \notin \overline{\{t\}}$ can be separated by disjoint neighbourhoods.

Dixmier 1968 — $\mathrm{Sep}(A)$, the set of all separated points, dense G_δ subset of $\mathrm{Prim}(A)$ as well as a Polish space;

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$\exists$ dense G_δ subset $S \subseteq X$ consisting of closed separated points which is a Polish space;

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S perfect, metrisable $\implies$ S not extremally disconnected.

Outline of proof

a topological lemma working in the background:

Lemma

Let X be a topological space, and let $G \subseteq X$ be a dense subset consisting of closed points.

(i) *If X is perfect then G is perfect (in itself).*

(ii) *For each $V \subseteq X$ open, $\overline{V} \cap G = \overline{V \cap G}^{G}$, where $^{-G}$ denotes the closure relative to G.*

(iii) *For each $V \subseteq X$ open, $\partial\big(\overline{V \cap G}^{G}\big) = \partial\overline{V} \cap G$.*

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let $\{V'_n \mid n \in \mathbb{N}\}$ be a countable basis for the topology of X;
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put $W_n = X \setminus \overline{V_n}$; then $O_n = V_n \cup W_n$ is dense open.

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let z_n, $n \in \mathbb{N}$ denote the equivalence class of $\chi_{V_n} \otimes 1 \in Z\big(M(C(O_n) \otimes B)\big)$ in $Z = Z(M_{\mathrm{loc}}(A))$;

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let $(e_n)_{n\in\mathbb{N}}$ be a strictly increasing approximate identity of B with $e_n e_{n+1} = e_n$ and $\|e_{n+1} - e_n\| = 1$ for all n;

put $p_1 = e_1$, $p_n = e_n - e_{n-1}$ for $n \geq 2$;

Outline of proof

setting $q_n = \sum_{j=1}^{n} z_j \otimes p_{2j}$, $n \in \mathbb{N}$, we obtain an increasing sequence $(q_n)_{n\in\mathbb{N}}$ in $M_{\mathrm{loc}}(A)_+$ bounded by 1.

$I(A)$ monotone complete $\implies$
$q = \sup_n q_n = \sum_{n=1}^{\infty} z_n \otimes p_{2n}$ exists in $I(A)_+$ and has norm 1.

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It remains to show

(a) $q \in M(K_A)$;

(b) $q \notin M_{\mathrm{loc}}(A)$.

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$$p_{2j}e_k = (e_{2j} - e_{2j-1})e_k = 0 \quad \text{if } 2j > k + 1.$$

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$\implies \forall\, a \in A,\ qa \in {}^cA \subseteq K_A \implies q \in M(K_A)$ by Lemma 3.

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towards (b) assume that $q \in M_{\rm loc}(A)$; hence

$\forall\, 0 < \varepsilon < 1/4 \;\exists\, U \subseteq X$ dense open, $m \in M(A(U))_{+,1}$: $\|m - q\| < \varepsilon$.

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$\forall\, 0 < \varepsilon < 1/4 \;\exists\, U \subseteq X$ dense open, $m \in M(A(U))_{+,1}$: $\|m - q\| < \varepsilon$.

take $n \in \mathbb{N}$ with $V_n' \subseteq U$ and choose $t_0 \in \partial\overline{V_n} \cap S \subseteq U \cap S$;
since t_0 can be approximated from 'inside' and 'outside' of V_n
and since S consists of separated points, the function $f(t) = \|ama + t\|$, $t \in U$ is continuous (some well-chosen $a \in A(U)$)
and attains both a value $> 1/2$ **and** $< 1/2$ at t_0, since

$$\left|f(t) - \chi_{V_n}(t)\right| \leq \|m - q\| + \varepsilon < 2\,\varepsilon;$$

Outline of proof

$I(A)$ monotone complete $\implies$
$q = \sup_n q_n = \sum_{n=1}^{\infty} z_n \otimes p_{2n}$ exists in $I(A)_+$ and has norm 1.

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Contradiction!

New Formulas for $M_{\mathrm{loc}}(A)$ and $I(A)$

A C^*-algebra

$$M_{\mathrm{loc}}(A) = \operatorname{alg}\varinjlim_{T\in\mathcal{T}} \Gamma_b(T, \mathsf{A}_{\mathfrak{M}_A})$$

$$I(A) = \operatorname{alg}\varinjlim_{T\in\mathcal{T}} \Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$$

where $\mathsf{A}_{\mathfrak{M}_A}$ and $\mathsf{A}_{\mathfrak{I}_A}$ are the upper semicontinuous C^*-bundles associated to the multiplier sheaf $\mathfrak{M}_A$ and the injective envelope sheaf $\mathfrak{I}_A$ of A, respectively;

$\mathcal{T}$ is the downwards directed family of dense G_δ subsets of $\mathrm{Prim}(A)$;

$\Gamma_b(T, -)$ denotes the bounded continuous local sections on T.

P. Ara, M. Mathieu, *Sheaves of C*-algebras*, Math. Nachrichten **283** (2010), 21–39.

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these descriptions are compatible: $\mathsf{A}_{\mathfrak{M}_A} \hookrightarrow \mathsf{A}_{\mathfrak{I}_A}$

Consequence:

$y \in M_{\rm loc}(M_{\rm loc}(A)) \subseteq I(A)$ is contained in some C^*-subalgebra $\Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$ and will belong to $M_{\rm loc}(A)$ once we find $T' \subseteq T$, $T' \in \mathcal{T}$ such that $y \in \Gamma_b(T', \mathsf{A}_{\mathfrak{M}_A})$.

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to be continued . . .