Local Multipliers and Derivations, Sheaves of C-Algebras and Cohomology*

Martin Mathieu

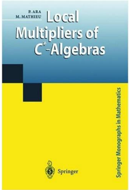
(Queen's University Belfast)

Shiraz, 27 April 2017

Partially supported by UK *Engineering and Physical Sciences Research Council* Grant No. EP/M02461X/1.

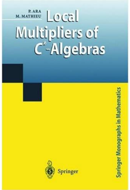
Part II: Operator theory via local multipliers

joint work with Pere Ara (Barcelona)



P. Ara and M. Mathieu, Local multipliers of C^*-algebras, Springer-Verlag, London, 2003.

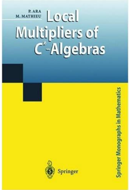
joint work with Pere Ara (Barcelona)



P. Ara and M. Mathieu, Local multipliers of C^*-algebras, Springer-Verlag, London, 2003.

- Automorphisms
- Derivations
- Elementary Operators
- Jordan Homomorphisms
- Lie Derivations, Lie Isomorphisms
- Centralising and Commuting Mappings
- Bi-derivations
- Commutativity Preserving Mapping

joint work with Pere Ara (Barcelona)



P. ARA AND M. MATHIEU, Local multipliers of C^*-algebras, Springer-Verlag, London, 2003.

Our **algebraic approach** to $M_{\mathrm{loc}}(A)$ enables us to solve complicated operator equations, e.g.,

$$\Big(\big([x,z]y[z,q(x)]-[z,q(x)]y[x,z]\big)r\big([x^2,z]y[x,z]-[x,z]y[x^2,z]\big)$$
$$-\big([x^2,z]y[x,z]-[x,z]y[x^2,z]\big)r\big([x,z]y[z,q(x)]-[z,q(x)]y[x,z]\big)\Big)\times$$
$$\times\, u\left([w^2,v]t[w,v]-[w,v]t[w^2,v]\right)=0$$

for fixed $x,y,z\in A$ and all $r,t,u,v,w\in A$.

1978 Pedersen introduces $M_{\mathrm{loc}}(A)$

Theorem

Let A be a separable C-algebra. Every derivation $d\colon A \to A$ extends uniquely to a derivation $d\colon M_{\mathrm{loc}}(A) \to M_{\mathrm{loc}}(A)$ and there is $y \in M_{\mathrm{loc}}(A)$ such that $d = \operatorname{ad} y$ (that is, $dx = [x, y] = xy - yx$ for all $x \in A$).*

1978 Pedersen introduces $M_{\text{loc}}(A)$

Theorem

Let A be a separable C-algebra. Every derivation $d\colon A \to A$ extends uniquely to a derivation $d\colon M_{\text{loc}}(A) \to M_{\text{loc}}(A)$ and there is $y \in M_{\text{loc}}(A)$ such that $d = \operatorname{ad} y$ (that is, $dx = [x, y] = xy - yx$ for all $x \in A$).*

Question

Is every derivation $d\colon M_{\text{loc}}(A) \to M_{\text{loc}}(A)$ inner?

Inner derivations on C^*-algebras

Inner derivations on C^*-algebras

Theorem (Kaplansky 1953)

Every $d \in Der(B(H))$ is inner.

Theorem (Kadison–Sakai 1966)

Every $d \in Der(A)$, A a von Neumann algebra, is inner.

Theorem (Sakai 1970)

Every $d \in Der(A)$, A a unital simple C^*-algebra, is inner.

Theorem (Sakai 1971)

Every $d \in Der(A)$, A a simple C^*-algebra, is inner in $M(A)$.

Inner derivations on C^*-algebras

Theorem (Akemann–Elliott–Pedersen–Tomiyama 1976/1979)

Let A be a separable C^*-algebra. Every derivation $d\colon A \to A$ is inner in $M(A)$ if and only if A is the direct sum of a continuous trace C^*-algebra and a C^*-algebra with discrete spectrum.

Inner derivations on C^*-algebras

Theorem (Pedersen 1978)

Let A be a separable C^*-algebra. Every derivation $d\colon A \to A$ extends uniquely to a derivation $d\colon M_{\mathrm{loc}}(A) \to M_{\mathrm{loc}}(A)$ and there is $y \in M_{\mathrm{loc}}(A)$ such that $d = \operatorname{ad} y$.

Inner derivations on C^*-algebras

Theorem (Pedersen 1978)

Let A be a separable C^*-algebra. Every derivation $d\colon A \to A$ extends uniquely to a derivation $d\colon M_{\rm loc}(A) \to M_{\rm loc}(A)$ and there is $y \in M_{\rm loc}(A)$ such that $d = \operatorname{ad} y$.

Theorem (Ara–Mathieu 2011)

Let A be a quasi-central separable C^*-algebra such that $\mathrm{Prim}(A)$ contains a dense G_δ subset consisting of closed points. Then every derivation of $M_{\rm loc}(A)$ is inner.

Outline of the argument

let $d\colon M_{\mathrm{loc}}(A) \to M_{\mathrm{loc}}(A)$, let $A \subseteq B \subseteq M_{\mathrm{loc}}(A)$ separable C^*-subalgebra such that $dB \subseteq B$;
extend $d_{|B}$ uniquely to $d_{M_{\mathrm{loc}}(B)}\colon M_{\mathrm{loc}}(B) \to M_{\mathrm{loc}}(B)$;

Outline of the argument

let $d\colon M_{\mathrm{loc}}(A) \to M_{\mathrm{loc}}(A)$, let $A \subseteq B \subseteq M_{\mathrm{loc}}(A)$ separable C^*-subalgebra such that $dB \subseteq B$;
extend $d_{|B}$ uniquely to $d_{M_{\mathrm{loc}}(B)}\colon M_{\mathrm{loc}}(B) \to M_{\mathrm{loc}}(B)$;

next extend both these derivations to the respective injective envelopes, but since
$I(B) = I(M_{\mathrm{loc}}(B))$ we have $d_{I(B)} = d_{I(M_{\mathrm{loc}}(B))}$;

Outline of the argument

let $d\colon M_{\mathrm{loc}}(A) \to M_{\mathrm{loc}}(A)$, let $A \subseteq B \subseteq M_{\mathrm{loc}}(A)$ separable C^*-subalgebra such that $dB \subseteq B$;
extend $d_{|B}$ uniquely to $d_{M_{\mathrm{loc}}(B)}\colon M_{\mathrm{loc}}(B) \to M_{\mathrm{loc}}(B)$;

next extend both these derivations to the respective injective envelopes, but since
$I(B) = I(M_{\mathrm{loc}}(B))$ we have $d_{I(B)} = d_{I(M_{\mathrm{loc}}(B))}$;

now extend d to $I(M_{\mathrm{loc}}(A))$; since $I(B) = I(A) = I(M_{\mathrm{loc}}(A))$,

$$d_{I(M_{\mathrm{loc}}(A))} = d_{I(B)} = d_{I(M_{\mathrm{loc}}(B))}.$$

Outline of the argument

let $d\colon M_{\rm loc}(A) \to M_{\rm loc}(A)$, let $A \subseteq B \subseteq M_{\rm loc}(A)$ separable C^*-subalgebra such that $dB \subseteq B$;
extend $d_{|B}$ uniquely to $d_{M_{\rm loc}(B)}\colon M_{\rm loc}(B) \to M_{\rm loc}(B)$;

next extend both these derivations to the respective injective envelopes, but since
$I(B) = I(M_{\rm loc}(B))$ we have $d_{I(B)} = d_{I(M_{\rm loc}(B))}$;

now extend d to $I(M_{\rm loc}(A))$; since $I(B) = I(A) = I(M_{\rm loc}(A))$,

$$d_{I(M_{\rm loc}(A))} = d_{I(B)} = d_{I(M_{\rm loc}(B))}.$$

$\overset{\text{Pedersen}}{\Longrightarrow}$ $d_{M_{\rm loc}(B)} = d_y$ some $y \in M_{\rm loc}(B) \overset{\text{our theorem}}{\subseteq} M_{\rm loc}(A)$;

consequently, $d = d_y$ on $M_{\rm loc}(A)$. $\square$

Every derivation on $M_{\mathrm{loc}}(A)$ is inner if

(i) $M_{\mathrm{loc}}(A) = A$ and every derivation on A is inner:
- A von Neumann algebra (Kadison–Sakai);
- A AW^*-algebra (Olesen);
- A simple unital (Sakai).

(ii) $M_{\mathrm{loc}}(A) = M(A)$ and every derivation on A is inner in $M(A)$:
- A simple (Sakai).

(iii) $M_{\mathrm{loc}}(A)$ simple (possible by Ara–Mathieu 1999!)

(iv) $M_{\mathrm{loc}}(A)$ AW^*-algebra:
- A commutative;
- A unital separable type I (Somerset 2000);
- A with all irreducible representations finite dimensional (Gogič 2013);

Every derivation on $M_{\mathrm{loc}}(A)$ is inner if

(i) $M_{\mathrm{loc}}(A) = A$ and every derivation on A is inner:
- A von Neumann algebra (Kadison–Sakai);
- A AW^*-algebra (Olesen);
- A simple unital (Sakai).

(ii) $M_{\mathrm{loc}}(A) = M(A)$ and every derivation on A is inner in $M(A)$:
- A simple (Sakai).

(iii) $M_{\mathrm{loc}}(A)$ simple (possible by Ara–Mathieu 1999!)

(iv) $M_{\mathrm{loc}}(A)$ AW^*-algebra:
- A commutative;
- A unital separable type I (Somerset 2000);
- A with all irreducible representations finite dimensional (Gogič 2013);

in all these cases $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$

Summary

- we have no example in which $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$ and we do not know that every derivation of $M_{\mathrm{loc}}(A)$ is inner;
- we have no example in which $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \neq M_{\mathrm{loc}}(A)$ and we know every derivation of $M_{\mathrm{loc}}(A)$ is inner.

Inner derivations on C^*-algebras

Theorem (Ara–Mathieu 2011)

Let A be a quasi-central separable C^*-algebra such that $\mathrm{Prim}(A)$ contains a dense G_δ subset consisting of closed points. Then every derivation of $M_{\mathrm{loc}}(A)$ is inner.

Theorem (Ara–Mathieu 2011)

Let A be a quasi-central separable C^*-algebra such that $\mathrm{Prim}(A)$ contains a dense G_δ subset consisting of closed points. Let D be a C^*-subalgebra of $M_{\mathrm{loc}}(A)$ containing A. Then $M_{\mathrm{loc}}(D) \subseteq M_{\mathrm{loc}}(A)$. In particular, $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$.

new tool: *a sheaf theory for general C^*-algebras*

A sufficient condition

A quasi-central if no primitive ideal of A contains $Z(A)$;

e.g., A unital or A commutative

B simple; B quasi-central $\iff$ B unital.

A sufficient condition

Theorem (Ara–Mathieu 2011)

Let A be a quasi-central separable C-algebra such that $\mathrm{Prim}(A)$ contains a dense G_δ subset consisting of closed points. Let D be a C*-subalgebra of $M_{\mathrm{loc}}(A)$ containing A. Then $M_{\mathrm{loc}}(D) \subseteq M_{\mathrm{loc}}(A)$. In particular, $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$.*

A quasi-central if no primitive ideal of A contains $Z(A)$;

e.g., A unital or A commutative

B simple; B quasi-central $\iff$ B unital.

New Formulas for $M_{\mathrm{loc}}(A)$ and $I(A)$

A C^*-algebra

$$M_{\mathrm{loc}}(A) = \operatorname{alg}\varinjlim_{T\in\mathcal{T}} \Gamma_b(T, \mathsf{A}_{\mathfrak{M}_A})$$

$$I(A) = \operatorname{alg}\varinjlim_{T\in\mathcal{T}} \Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$$

where $\mathsf{A}_{\mathfrak{M}_A}$ and $\mathsf{A}_{\mathfrak{I}_A}$ are the upper semicontinuous C^*-bundles associated to the multiplier sheaf $\mathfrak{M}_A$ and the injective envelope sheaf $\mathfrak{I}_A$ of A, respectively;

$\mathcal{T}$ is the downwards directed family of dense G_δ subsets of $\mathrm{Prim}(A)$;

$\Gamma_b(T, -)$ denotes the bounded continuous local sections on T.

P. Ara, M. Mathieu, *Sheaves of C*-algebras*, Math. Nachrichten **283** (2010), 21–39.

New Formulas for $M_{\mathrm{loc}}(A)$ and $I(A)$

A C^*-algebra

$$M_{\mathrm{loc}}(A) = \operatorname{alg}\varinjlim{}_{T\in\mathcal{T}}\, \Gamma_b(T, \mathsf{A}_{\mathfrak{M}_A})$$

$$I(A) = \operatorname{alg}\varinjlim{}_{T\in\mathcal{T}}\, \Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$$

these descriptions are compatible: $\mathsf{A}_{\mathfrak{M}_A} \hookrightarrow \mathsf{A}_{\mathfrak{I}_A}$

Consequence:

$y \in M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \subseteq I(A)$ is contained in some C^*-subalgebra $\Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$ and will belong to $M_{\mathrm{loc}}(A)$ once we find $T' \subseteq T$, $T' \in \mathcal{T}$ such that $y \in \Gamma_b(T', \mathsf{A}_{\mathfrak{M}_A})$.

P. Ara, M. Mathieu, *Sheaves of C*-algebras*, Math. Nachrichten **283** (2010), 21–39.

Sheaves of C^*-algebras

X a topological space;
$\mathcal{O}_X$ category of open subsets (with open subsets U as objects and $V \to U$ if and only if $V \subseteq U$).

$\mathcal{C}^*$ category of C^*-algebras.

Definition

A *presheaf of C^*-algebras* is a contravariant functor $\mathfrak{A} \colon \mathcal{O}_X \to \mathcal{C}^*$.

A *sheaf of C^*-algebras* is a presheaf $\mathfrak{A}$ such that $\mathfrak{A}(\emptyset) = 0$ and, for every open subset U of X and every open cover $U = \bigcup_i U_i$, the maps $\mathfrak{A}(U) \to \mathfrak{A}(U_i)$ are the limit of the diagrams $\mathfrak{A}(U_i) \to \mathfrak{A}(U_i \cap U_j)$ for all i, j.

Sheaves of C^*-algebras

Notation and Terminology:

the C^*-algebra $\mathfrak{A}(U)$ is the *section algebra* over $U \in \mathcal{O}_X$;

by $s_{|V}$, $V \subseteq U$ open, we mean the "restriction" of $s \in \mathfrak{A}(U)$ to V; i.e., the image of s in $\mathfrak{A}(V)$ under $\mathfrak{A}(U) \to \mathfrak{A}(V)$;

the unique gluing property of a sheaf can be expressed as follows:

for each compatible family of sections $s_i \in \mathfrak{A}(U_i)$, i.e., $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for all i, j, there is a unique section $s \in \mathfrak{A}(U)$ such that $s_{|U_i} = s_i$ for all i.

Sheaves of C^*-algebras

Example 1. *Sheaves from bundles*

Let (A, π, X) be an upper semicontinuous C^*-bundle. Then

$$\Gamma_b(-, \mathsf{A}) \colon \mathcal{O}_X \to \mathcal{C}_1^*, \quad U \mapsto \Gamma_b(U, \mathsf{A})$$

defines the *sheaf of bounded continuous local sections of* A, where $\mathcal{C}_1^*$ is the category of unital C^*-algebras.

$\Gamma_b(U, \mathsf{A}) \to \Gamma_b(V, \mathsf{A})$, $V \subseteq U$, is the usual restriction map.

Sheaves of C^*-algebras

Example 2. *The multiplier sheaf*

A C^*-algebra with primitive ideal space $\mathrm{Prim}(A)$;

$$\mathfrak{M}_A \colon \mathcal{O}_{\mathrm{Prim}(A)} \to \mathcal{C}_1^*, \quad \mathfrak{M}_A(U) = M(A(U)),$$

where $M(A(U))$ denotes the multiplier algebra of the closed ideal $A(U)$ of A associated to the open subset $U \subseteq \mathrm{Prim}(A)$.
$M(A(U)) \to M(A(V))$, $V \subseteq U$, the restriction homomorphisms.

Sheaves of C^*-algebras

Example 2. *The multiplier sheaf*

A C^*-algebra with primitive ideal space $\mathrm{Prim}(A)$;

$$\mathfrak{M}_A \colon \mathcal{O}_{\mathrm{Prim}(A)} \to \mathcal{C}_1^*, \quad \mathfrak{M}_A(U) = M(A(U)),$$

where $M(A(U))$ denotes the multiplier algebra of the closed ideal $A(U)$ of A associated to the open subset $U \subseteq \mathrm{Prim}(A)$.
$M(A(U)) \to M(A(V))$, $V \subseteq U$, the restriction homomorphisms.

Proposition

The above functor $\mathfrak{M}_A$ defines a sheaf of C^*-algebras.

Sheaves of C^*-algebras

Example 3. *The injective envelope sheaf*

let $I(B)$ denote the *injective envelope* of B;

$$\mathfrak{I}_A\colon \mathcal{O}_{\mathrm{Prim}(A)} \to \mathcal{C}_1^*, \quad \mathfrak{I}_A(U) = p_U I(A) = I(A(U)),$$

where $p_U = p_{A(U)}$ denotes the unique central open projection in $I(A)$ such that $p_{A(U)} I(A)$ is the injective envelope of $A(U)$. $I(A(U)) \to I(A(V))$, $V \subseteq U$, given by multiplication by p_V (as $p_V \leq p_U$).

$\{p_U \mid U \in \mathcal{O}_{\mathrm{Prim}(A)}\}$ is a complete Boolean algebra isomorphic to the Boolean algebra of regular open subsets of $\mathrm{Prim}(A)$, and it is precisely the set of projections of the AW^*-algebra $Z(I(A))$.

Bundles of C^*-algebras

Definition

For a topological space X, an *upper semicontinuous C^*-bundle over X* (in short, a *usc C^*-bundle over X*) is a triple (A, π, X) consisting of a topological space A and an open, continuous surjection $\pi\colon \mathsf{A} \to X$ with each fibre $\mathsf{A}_x := \pi^{-1}(x)$ a C^*-algebra and such that the function $\|\cdot\|\colon \mathsf{A} \to \mathbb{R}$ defined by $a \mapsto \|a\|_{\mathsf{A}_{\pi(a)}}$ is upper semicontinuous and all algebraic operations are continuous on A;

Bundles of C^*-algebras

Definition

For a topological space X, an *upper semicontinuous C^*-bundle over X* (in short, a *usc C^*-bundle over X*) is a triple (A, π, X) consisting of a topological space A and an open, continuous surjection $\pi\colon \mathsf{A} \to X$ with each fibre $\mathsf{A}_x := \pi^{-1}(x)$ a C^*-algebra and such that the function $\|\cdot\|\colon \mathsf{A} \to \mathbb{R}$ defined by $a \mapsto \|a\|_{\mathsf{A}_{\pi(a)}}$ is upper semicontinuous and all algebraic operations are continuous on A;
that is, $+$ and $\cdot$ are continuous functions $\mathsf{A} \times_\pi \mathsf{A} \to \mathsf{A}$ (where $\mathsf{A} \times_\pi \mathsf{A} = \{(a_1, a_2) \in \mathsf{A} \times \mathsf{A} \mid \pi(a_1) = \pi(a_2)\}$) and ${}^*\colon \mathsf{A} \to \mathsf{A}$ as well as $\cdot_{\mathbb{C}}\colon \mathbb{C} \times \mathsf{A} \to \mathsf{A}$ are continuous.

Bundles of C^*-algebras

Definition (ctd.)

Denoting by $\Gamma_b(U, \mathrm{A})$, $U \in \mathcal{O}_X$ the set of all bounded continuous sections $s\colon U \to \mathrm{A}$ of π we further require the following properties.

(i) For all $U \in \mathcal{O}_X$, $s \in \Gamma_b(U, \mathrm{A})$ and $\varepsilon > 0$, the set

$$V(U, s, \varepsilon) := \{a \in \mathrm{A} \mid \pi(a) \in U \text{ and } \|a - s(\pi(a))\| < \varepsilon\}$$

is an open subset of A and these sets form a basis for the topology of A.

(ii) For each $x \in X$, we have

$$\mathrm{A}_x = \overline{\{s(x) \mid s \in \Gamma_b(U, \mathrm{A}),\ U \text{ an open neighbourhood of } x\}}.$$

Bundles of C^*-algebras

Example

$A = C(X, B(H))$ yields a trivial continuous C^*-bundle over the compact Hausdorff space X with each fibre equal to $B(H)$.

Bundles of C^*-algebras

Example

$A = C(X, B(H))$ yields a trivial continuous C^*-bundle over the compact Hausdorff space X with each fibre equal to $B(H)$.

Example (Somerset)

For a separable unital C^*-algebra A, $M_{\mathrm{loc}}(A)$ can be realised as a continuous C^*-bundle over $\mathrm{Glimm}(M_{\mathrm{loc}}(A)) = \beta\,\mathrm{Prim}(M_{\mathrm{loc}}(A))$, the Glimm ideal space of $M_{\mathrm{loc}}(A)$, with all fibres being primitive C^*-algebras.

Bundles of C^*-algebras

X a locally compact Hausdorff space

Definition

A C^*-algebra A is a *$C_0(X)$-algebra* if there is an essential *-homomorphism $\iota\colon C_0(X) \to ZM(A)$ (i.e., $\overline{\iota(C_0(X))A} = A$).

Bundles of C^*-algebras

X a locally compact Hausdorff space

Definition

A C^*-algebra A is a *$C_0(X)$-algebra* if there is an essential *-homomorphism $\iota\colon C_0(X) \to ZM(A)$ (i.e., $\overline{\iota(C_0(X))A} = A$).

Definition

A *C^*-algebra over X* is a pair (A, ψ) consisting of a C^*-algebra A and a continuous mapping $\psi\colon \mathrm{Prim}(A) \to X$.

Bundles of C^*-algebras

X a locally compact Hausdorff space

Theorem (Fell, Lee)

For a C^*-algebra A, the following conditions are equivalent:

(a) *A is a $C_0(X)$-algebra;*

(b) *(A, ψ) is a C^*-algebra over X;*

(c) *A is the section algebra of a usc C^*-bundle (A, π, X) (that is, there is a $C_0(X)$-linear isomorphism from A onto $\Gamma_0(X)$).*

Moreover, (A, π, X) is a continuous C^*-bundle if and only if $\psi\colon \mathrm{Prim}(A) \to X$ is open.

from bundles to sheaves

$$(\mathsf{A}, \pi, X)$$

$$\Gamma_b(-, \mathsf{A})$$

$$\mathfrak{A}\colon \mathcal{O}_X \to \mathcal{C}^*$$

from bundles to sheaves

$$(\mathsf{A},\pi,X) \longrightarrow \Gamma_b(-,\mathsf{A})$$

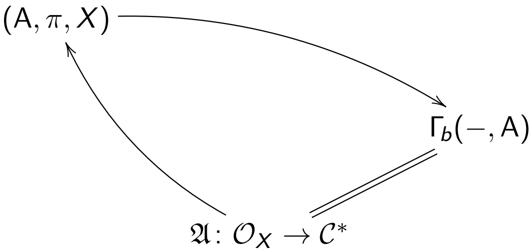
$$\mathfrak{A}\colon \mathcal{O}_X \to \mathcal{C}^*$$

from bundles to sheaves

$$(\mathsf{A}, \pi, X) \longrightarrow \Gamma_b(-, \mathsf{A})$$

$$\Gamma_b(-, \mathsf{A}) = \mathfrak{A}\colon \mathcal{O}_X \to \mathcal{C}^*$$

from bundles to sheaves



from sheaves to bundles

$$(\mathsf{A}, \pi, X)$$

$$\Gamma_b(-, \mathsf{A})$$

$$\mathfrak{A} \colon \mathcal{O}_X \to \mathcal{C}^*$$

from sheaves to bundles

(A, π, X)

$\Gamma_b(-, \mathsf{A})$

$\mathfrak{A}\colon \mathcal{O}_X \to \mathcal{C}^*$

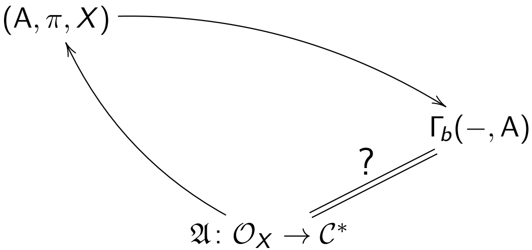
from sheaves to bundles

(A, π, X)

$\Gamma_b(-, \mathsf{A})$

$\mathfrak{A}\colon \mathcal{O}_X \to \mathcal{C}^*$

from sheaves to bundles and back?



from sheaves to bundles

Theorem

Given a presheaf $\mathfrak{A}$ of C-algebras over X, there is a canonically associated upper semicontinuous C*-bundle (A, π, X) over X.*

from sheaves to bundles

Theorem

Given a presheaf $\mathfrak{A}$ of C-algebras over X, there is a canonically associated upper semicontinuous C*-bundle (A, π, X) over X.*

Idea:

$x \in X$, define $\mathrm{A}_x := \varinjlim_{x \in U} \mathfrak{A}(U)$ (stalk at x)

let $\mathrm{A} := \bigsqcup_{x \in X} \mathrm{A}_x$ and define a topology on A by

$$V(U, s, \varepsilon) = \{a \in \mathrm{A} \mid \pi(a) \in U \text{ and } \|a - s(\pi(a))\| < \varepsilon\}$$

is a basic open set, where $\varepsilon > 0$, $U \in \mathcal{O}_X$, $s \in \mathfrak{A}(U)$ and $s(x)$ the image under $\mathfrak{A}(U) \to \mathrm{A}_x$.

The local multiplier sheaf

Definition

For a C^*-algebra A define the local multiplier sheaf $\mathfrak{Mloc}_A$ by

$$\mathfrak{Mloc}_A(U) = M_{\mathrm{loc}}(A(U)) = p_U M_{\mathrm{loc}}(A) \quad (U \in \mathcal{O}_{\mathrm{Prim}(A)}),$$

where $M_{\mathrm{loc}}(A) \subseteq I(A)$ and $p_U \in Z(M_{\mathrm{loc}}(A)) = Z(I(A))$.

note: $\mathfrak{M}_A \hookrightarrow \mathfrak{Mloc}_A \hookrightarrow \mathfrak{I}_A$ as sheaves

The local multiplier sheaf

Definition

For a C^*-algebra A define the local multiplier sheaf $\mathfrak{Mloc}_A$ by

$$\mathfrak{Mloc}_A(U) = M_{\rm loc}(A(U)) = p_U M_{\rm loc}(A) \quad (U \in \mathcal{O}_{\mathrm{Prim}(A)}),$$

where $M_{\rm loc}(A) \subseteq I(A)$ and $p_U \in Z(M_{\rm loc}(A)) = Z(I(A))$.

note: $\mathfrak{M}_A \hookrightarrow \mathfrak{Mloc}_A \hookrightarrow \mathfrak{I}_A$ as sheaves

aim: *a sheaf representation of* $M_{\rm loc}(A)$

The derived sheaf of a presheaf

X Baire space (e.g., $X = \mathrm{Prim}(A)$)

$\mathcal{T}$ the family of dense G_δ's of X

(A, π, X) an upper semicontinuous C^*-bundle

The derived sheaf of a presheaf

X Baire space (e.g., $X = \mathrm{Prim}(A)$)

$\mathcal{T}$ the family of dense G_δ's of X

(A, π, X) an upper semicontinuous C^*-bundle

$U \in \mathcal{O}_X$: $\quad \mathfrak{D}(U) = \mathrm{alg}\varinjlim_{T \in \mathcal{T}} \Gamma_b(T \cap U, \mathsf{A})$

$T' \subseteq T \in \mathcal{T}$: $\quad \Gamma_b(T \cap U, \mathsf{A}) \to \Gamma_b(T' \cap U, \mathsf{A})$ restriction maps

The derived sheaf of a presheaf

X Baire space (e.g., $X = \mathrm{Prim}(A)$)

$\mathcal{T}$ the family of dense G_δ's of X

(A, π, X) an upper semicontinuous C^*-bundle

$U \in \mathcal{O}_X$: $\qquad \mathfrak{D}(U) = \mathrm{alg}\varinjlim_{T\in\mathcal{T}} \Gamma_b(T \cap U, \mathsf{A})$

$T' \subseteq T \in \mathcal{T}$: $\qquad \Gamma_b(T \cap U, \mathsf{A}) \to \Gamma_b(T' \cap U, \mathsf{A})$ restriction maps

Proposition

$\mathfrak{D} = \mathfrak{D}_{(\mathsf{A},\pi,X)}$ *is a presheaf of C^*-algebras over X.*

The derived sheaf of a presheaf

Definition

Let $\mathfrak{A}$ be a presheaf of C^*-algebras over a Baire space X. The *derived presheaf* $\mathfrak{D}_{\mathfrak{A}}$ of $\mathfrak{A}$ is the presheaf $\mathfrak{D}_{(\mathsf{A},\pi,X)}$.

The derived sheaf of a presheaf

Definition

Let $\mathfrak{A}$ be a presheaf of C^*-algebras over a Baire space X. The *derived presheaf* $\mathfrak{D}_{\mathfrak{A}}$ of $\mathfrak{A}$ is the presheaf $\mathfrak{D}_{(\mathsf{A},\pi,X)}$.

Theorem

Let X be a Baire space. The map $\mathfrak{D}$ defines a functor

$$\mathfrak{D}\colon \mathcal{PSh}(X,\mathcal{C}_1^*) \longrightarrow \mathcal{Sh}(X,\mathcal{C}_1^*).$$

If $\iota\colon \mathfrak{A} \to \mathfrak{B}$ is a faithful natural transformation (that is, $\iota_U\colon \mathfrak{A}(U) \to \mathfrak{B}(U)$ is injective for every $U \in \mathcal{O}_X$), then $\mathfrak{D}(\iota)\colon \mathfrak{D}_{\mathfrak{A}} \to \mathfrak{D}_{\mathfrak{B}}$ is also faithful.

P. Ara, M. Mathieu, *Sheaves of C*-algebras*, Math. Nachrichten **283** (2010), 21–39.

The derived sheaf of a presheaf

Theorem

For every C-algebra A, we have*

$$\mathfrak{D}_{\mathfrak{M}_A} \cong \mathfrak{Mloc}_A \quad \textit{and} \quad \mathfrak{D}_{\mathfrak{I}_A} \cong \mathfrak{I}_A$$

as sheaves over $\mathrm{Prim}(A)$.

hence

$$\mathfrak{Mloc}_A(U) = \operatorname{alg}\varinjlim{}_{T\in\mathcal{T}} \Gamma_b(U \cap T, \mathsf{A}_{\mathfrak{M}_A}) \hookrightarrow \operatorname{alg}\varinjlim{}_{T\in\mathcal{T}} \Gamma_b(U \cap T, \mathsf{A}_{\mathfrak{I}_A}) = \mathfrak{I}_A(U)$$

for each $U \in \mathcal{O}_{\mathrm{Prim}(A)}$.

Back to derivations

A C^*-algebra

$$M_{\mathrm{loc}}(A) = \operatorname{alg}\varinjlim_{T\in\mathcal{T}} \Gamma_b(T, \mathsf{A}_{\mathfrak{M}_A})$$

$$I(A) = \operatorname{alg}\varinjlim_{T\in\mathcal{T}} \Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$$

Theorem (simplified version)

Let A be a quasi-central separable C^*-algebra such that $\mathrm{Prim}(A)$ contains a dense G_δ subset consisting of closed points.
Then $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$.

Outline of proof

Outline of proof

take $y \in M(J)$ for some closed essential ideal J of $M_{\mathrm{loc}}(A)$;
let $T \in \mathcal{T}$ be such that $y \in \Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$;
WLOG T consists of closed separated points of $\mathrm{Prim}(A)$.

Outline of proof

take $y \in M(J)$ for some closed essential ideal J of $M_{\rm loc}(A)$;
let $T \in \mathcal{T}$ be such that $y \in \Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$;
WLOG T consists of closed separated points of $\mathrm{Prim}(A)$.

recall: $t \in \mathrm{Prim}(A)$ is *separated* if t and every point $t' \notin \overline{\{t\}}$ can be separated by disjoint neighbourhoods.

Dixmier 1968 $\mathrm{Sep}(A)$, the set of all separated points, dense G_δ subset of $\mathrm{Prim}(A)$ as well as a Polish space;

Outline of proof

take $y \in M(J)$ for some closed essential ideal J of $M_{\mathrm{loc}}(A)$;
let $T \in \mathcal{T}$ be such that $y \in \Gamma_b(T, \mathsf{A}_{\mathfrak{J}_A})$;
WLOG T consists of closed separated points of $\mathrm{Prim}(A)$.

Lemma: There is $h \in J$ such that $h(t) \neq 0$ for all $t \in T$.

Outline of proof

take $y \in M(J)$ for some closed essential ideal J of $M_{\mathrm{loc}}(A)$;
let $T \in \mathcal{T}$ be such that $y \in \Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$;
WLOG T consists of closed separated points of $\mathrm{Prim}(A)$.

Lemma: There is $h \in J$ such that $h(t) \neq 0$ for all $t \in T$.

Lemma: There is a separable C^*-subalgebra $B \subseteq J$ with $AhA \subseteq B$ and $y \in M(B)$.

Outline of proof

take $y \in M(J)$ for some closed essential ideal J of $M_{\mathrm{loc}}(A)$;
let $T \in \mathcal{T}$ be such that $y \in \Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$;
WLOG T consists of closed separated points of $\mathrm{Prim}(A)$.

Lemma: There is $h \in J$ such that $h(t) \neq 0$ for all $t \in T$.

Lemma: There is a separable C^*-subalgebra $B \subseteq J$ with $AhA \subseteq B$ and $y \in M(B)$.

take countable dense subset $\{b_n \mid n \in \mathbb{N}\}$ in B and $T_n \in \mathcal{T}$ such that $b_n \in \Gamma_b(T_n, \mathsf{A}_{\mathfrak{M}_A})$; put $\mathsf{A} = \mathsf{A}_{\mathfrak{M}_A}$;
letting $T' = \bigcap_n T_n \cap T \in \mathcal{T}$, we have $B \subseteq \Gamma_b(T', \mathsf{A})$, hence

$$B_t = \{b(t) \mid b \in B\} \subseteq \mathsf{A}_t \qquad (t \in T').$$

Outline of proof

in general, $\exists\ \varphi_t \colon \mathrm{A}_t \to M_{\mathrm{loc}}(A/t)$

$$\left.\begin{array}{rl} A \quad \text{quasicentral} \;\Rightarrow\; A/t \text{ unital} \\ t \;\; \text{closed} \;\Rightarrow\; A/t \text{ simple} \end{array}\right\} \Rightarrow \quad M_{\mathrm{loc}}(A/t) = A/t.$$

Outline of proof

in general, $\exists\ \varphi_t\colon \mathrm{A}_t \to M_{\mathrm{loc}}(A/t)$

$$\left.\begin{array}{rcl} A \;\; \text{quasicentral} & \Rightarrow & A/t \text{ unital} \\ t \;\; \text{closed} & \Rightarrow & A/t \text{ simple} \end{array}\right\} \Rightarrow \quad M_{\mathrm{loc}}(A/t) = A/t.$$

Main Lemma: A quasicentral, $t \in \mathrm{Prim}(A)$ closed, separated

$\Rightarrow\ \varphi_t$ isomorphism.

rests on existence of local identities in quasicentral C^*-algebras:

$$\forall\ t \in \mathrm{Prim}(A) \quad \exists\ U_1 \subseteq \mathrm{Prim}(A) \text{ open}, t \in U_1,$$
$$\exists\ z \in Z(A)_+,\ \|z\| = 1\colon\ z + A(U_2) = 1_{A/A(U_2)},$$

where $U_2 = \mathrm{Prim}(A) \setminus \overline{U_1}$.

Outline of proof

in general, $\exists\ \varphi_t\colon \mathsf{A}_t \to M_{\mathrm{loc}}(A/t)$

$$\left.\begin{array}{rl} A \ \text{ quasicentral} & \Rightarrow\ A/t \text{ unital} \\ t \ \text{ closed} & \Rightarrow\ A/t \text{ simple} \end{array}\right\} \Rightarrow\ M_{\mathrm{loc}}(A/t) = A/t.$$

Main Lemma: A quasicentral, $t \in \mathrm{Prim}(A)$ closed, separated $\Rightarrow\ \varphi_t$ isomorphism.

thus,

$$\mathsf{A}_t = \mathsf{A}_t\, h(t)\, \mathsf{A}_t = (A/t)h(t)(A/t) = (AhA)_t \subseteq B_t \subseteq \mathsf{A}_t \quad (t \in T').$$

Outline of proof

in general, $\exists\ \varphi_t\colon \mathsf{A}_t \to M_{\mathrm{loc}}(A/t)$

$$\left.\begin{array}{rl} A \quad \text{quasicentral} \;\Rightarrow\; A/t \text{ unital} \\ t \;\; \text{closed} \;\Rightarrow\; A/t \text{ simple} \end{array}\right\} \Rightarrow \quad M_{\mathrm{loc}}(A/t) = A/t.$$

Main Lemma: A quasicentral, $t \in \mathrm{Prim}(A)$ closed, separated

$\Rightarrow\ \varphi_t$ isomorphism.

thus,

$$\mathsf{A}_t = \mathsf{A}_t\, h(t)\, \mathsf{A}_t = (A/t)h(t)(A/t) = (AhA)_t \subseteq B_t \subseteq \mathsf{A}_t \quad (t \in T').$$

$$\Rightarrow \exists\ b_t \in B\colon\ b_t(t) = 1_{\mathsf{A}_t}$$
$$\Rightarrow y(t) = y(t)\, 1_{\mathsf{A}_t} = (yb_t)(t) \in \mathsf{A}_t \quad (t \in T').$$

Outline of proof

take $y \in M(J)$ for some closed essential ideal J of $M_{\mathrm{loc}}(A)$;
let $T \in \mathcal{T}$ be such that $y \in \Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$;
WLOG T consists of closed separated points of $\mathrm{Prim}(A)$.

it follows that $y \in \Gamma_b(T', \mathsf{A}_{\mathfrak{M}_A})$ with $T' \subseteq T$, proving that $y \in M_{\mathrm{loc}}(A)$. $\square$

Outline of proof

take $y \in M(J)$ for some closed essential ideal J of $M_{\mathrm{loc}}(A)$;
let $T \in \mathcal{T}$ be such that $y \in \Gamma_b(T, \mathsf{A}_{\mathfrak{I}_A})$;
WLOG T consists of closed separated points of $\mathrm{Prim}(A)$.

it follows that $y \in \Gamma_b(T', \mathsf{A}_{\mathfrak{M}_A})$ with $T' \subseteq T$, proving that $y \in M_{\mathrm{loc}}(A)$. $\square$

this nicely illustrates the usefulness of our sheaf theory