Local Multipliers and Derivations, Sheaves of C-Algebras and Cohomology*

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Shiraz, 27 April 2017

Partially supported by UK *Engineering and Physical Sciences Research Council* Grant No. EP/M02461X/1.

Part III: From Algebraic Geometry to Noncommutative Topology

. . . getting some guidance from Algebraic Geometry

Algebraic Geometry (in a nutshell)

"Algebraic Geometry is the study of the solutions of systems of polynomial equations in an affine or projective n-space with the aim to classify all algebraic varieties up to isomorphism."

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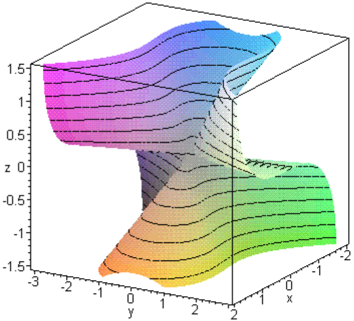

Algebraic Geometry (in a nutshell)

$\Bbbk$ algebraically closed field; $n \in \mathbb{N}$; $\mathbb{A}^n$ affine n-space

$X = Z(f_1, \ldots, f_\ell) \subseteq \mathbb{A}^n$ (affine algebraic) variety
(where $f_1, \ldots, f_\ell \in R = \Bbbk[x_1, \ldots, x_n]$)

$Y \subseteq \mathbb{A}^n$: $I(Y) = \{f \in R \mid f_{|Y} = 0\}$ ideal in R

$X = Z(I(X))$ and $I(X)$ prime ideal in R;

$A(X) = R/I(X)$ integral domain and a finitely generated $\Bbbk$-algebra

X, Y varieties: *morphism* $\varphi\colon X \to Y$ continuous s.t.

$$V \subseteq Y \text{ open, } f\colon V \to \Bbbk \text{ regular } \Rightarrow f \circ \varphi\colon \varphi^{-1}(V) \to \Bbbk \text{ regular}$$

("regular" = "can locally be written as the quotient of two polynomials")

Algebraic Geometry (in a nutshell)

X, Y varieties: $X \cong Y$ if and only if $A(X) \cong A(Y)$;

$\mathscr{V}\!ar_{\Bbbk}$ category of (affine) varieties over $\Bbbk$;

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Schemes

ringed space $(X, \mathfrak{O}_X)$,
X topological space, $\mathfrak{O}_X$ sheaf of rings on X

morphism $(f, f^{\#})\colon (X, \mathfrak{O}_X) \to (Y, \mathfrak{O}_Y)$
$f\colon X \to Y$ continuous
$f^{\#}\colon \mathfrak{O}_Y \to f_*\mathfrak{O}_X$ morphism of sheaves

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$(X, \mathfrak{O}_X)$ *locally ringed space* if stalk $(\mathfrak{O}_X)_t$ is local, $t \in X$

morphism $(f, f^{\#})$ s.t. $f_t^{\#}\colon (\mathfrak{O}_Y)_{f(t)} \to (\mathfrak{O}_X)_t$ is local, $t \in X$
(i.e., "fixes" the maximal ideal)

Algebraic Geometry (in a nutshell)

Schemes (ctd.)

A commutative unital ring, $\mathrm{Spec}(A)$ its prime ideal space

$p \in \mathrm{Spec}(A)$: $A_p = A[S_p^{-1}]$, $S_p = A \setminus p$ *localisation at* p

$U \subseteq \mathrm{Spec}(A)$ open: $\mathfrak{O}(U) \ni s \colon U \to \coprod_{p \in U} A_p$ s.t. $s(p) \in A_p$,

and "locally" $s = af^{-1}$;

$\mathfrak{O}(U)$ commutative unital ring,
$V \subseteq U \Rightarrow \text{rest} \colon \mathfrak{O}(U) \to \mathfrak{O}(V)$ unital homomorphism

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Fundamental Fact: $(\mathrm{Spec}(A), \mathfrak{O})$ locally ringed space

$\pi \colon A \to B$ unital homomorphism induces morphism

$$(f, f^{\#}) \colon (\mathrm{Spec}(B), \mathfrak{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathfrak{O}_{\mathrm{Spec}(A)})$$

and every morphism of locally ringed spaces arises in this way (via the global section functor).

Algebraic Geometry (in a nutshell)

Schemes (ctd.)

Definition (Grothendieck)

An affine scheme is a locally ringed space $(X, \mathfrak{O}_X)$ isomorphic to some $(\mathrm{Spec}(A), \mathfrak{O})$. A scheme is a locally ringed space $(X, \mathfrak{O}_X)$ which is *locally* an affine scheme. X is called the *topological space* of the scheme and $\mathfrak{O}_X$ its structure sheaf.

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Theorem (Grothendieck)

Letting $t(Y) = (\mathrm{Spec}(A(Y)), \mathfrak{O}_{\mathrm{Spec}(A(Y))})$ for any affine variety Y defines a fully faithful functor $\mathscr{V}\!ar_{\mathfrak{k}} \longrightarrow \mathscr{S}\!ch_{\mathfrak{k}}$ which yields a homeomorphism onto the set of closed points of $\mathrm{Spec}(A(Y))$ and the sheaf of regular functions of Y is obtained by restricting the structure sheaf of $t(Y)$ via this homeomorphism.

Algebraic Geometry (in a nutshell)

where do we go from here?

Algebraic Geometry (it turns out to be a coconut)

Algebraic Geometry: off to cohomology

Sheaves of modules $(X, \mathfrak{O}_X)$ ringed space,
sheaf of $\mathfrak{O}_X$-modules $\mathfrak{F}$ on X s.t. $\forall\, V \subseteq U \subseteq X$ open:
$\mathfrak{F}(U)$ is an $\mathfrak{O}_X(U)$-module and $\mathfrak{F}(U) \to \mathfrak{F}(V)$ compatible;

morphism
$\mathfrak{F} \to \mathfrak{G}$ morphism of sheaves s.t. $\mathfrak{F}(U) \to \mathfrak{G}(U)$ module map;

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Quasi-coherent sheaves $(X, \mathfrak{O}_X)$ scheme,
sheaf of $\mathfrak{O}_X$-modules $\mathfrak{F}$ *quasi-coherent* if
$\exists\ \{(A_i, M_i)\}$ s.t. $X = \bigcup_i \mathrm{Spec}(A_i)$ and $\mathfrak{F}_{|\mathrm{Spec}(A_i)} \cong \mathfrak{M}_i$
where, for (A, M),
$M_p = M \otimes_A A_p$, $p \in \mathrm{Spec}(A)$ and $U \subseteq \mathrm{Spec}(A)$ open:
$\mathfrak{M}(U) \ni s \colon U \to \coprod_{p \in U} M_p$ s.t. $s(p) \in M_p$, and "locally" $s = mf^{-1}$;

Algebraic Geometry: off to cohomology

Facts:

Let $(A, X = \mathrm{Spec}(A))$ be an affine scheme. Then

$$M \longmapsto \mathfrak{M}$$

gives an equivalence between A-$\mathscr{M}od$ and $\mathscr{Q}co(X)$, the category of quasi-coherent $\mathfrak{O}_X$-modules with inverse the global section functor.

Let $0 \longrightarrow \mathfrak{F}' \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{F}'' \longrightarrow 0$ be an exact sequence of $\mathfrak{O}_X$-modules such that $\mathfrak{F}'$ is quasi-coherent. Then

$$0 \longrightarrow \Gamma(X, \mathfrak{F}') \longrightarrow \Gamma(X, \mathfrak{F}) \longrightarrow \Gamma(X, \mathfrak{F}'') \longrightarrow 0$$

is exact.

Sheaf Cohomology (we are now on page 200 in [Hartshorne])

Sheaf Cohomology

$(X, \mathfrak{O}_X)$ ringed space, let $\mathscr{Mod}(X)$ be the category of $\mathfrak{O}_X$-modules

$$\cdots \longrightarrow \mathfrak{F}_{i-1} \xrightarrow{d^{i-1}} \mathfrak{F}_i \xrightarrow{d^i} \mathfrak{F}_{i+1} \longrightarrow \cdots$$

complex $\mathfrak{F}^\bullet$ in $\mathscr{Mod}(X)$ (i.e., $d^i \circ d^{i-1} = 0$);

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$i \in \mathbb{Z}$: $h^i(\mathfrak{F}^\bullet) = \ker d^i / \operatorname{im} d^{i-1}$ *homology*;

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a morphism of complexes $f \colon \mathfrak{F}^\bullet \to \mathfrak{G}^\bullet$ $(\{f^i \colon \mathfrak{F}_i \to \mathfrak{G}_i \mid i \in \mathbb{Z}\})$

yields $h^*(f) \colon h^*(\mathfrak{F}^\bullet) \to h^*(\mathfrak{G}^\bullet)$ so h^* is a covariant functor;

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$\mathfrak{F}^\bullet \sim \mathfrak{G}^\bullet$ (*"homotopic"*) $\Rightarrow$ $h^*(\mathfrak{F}^\bullet) \cong h^*(\mathfrak{G}^\bullet)$.

Sheaf Cohomology

$(X, \mathfrak{O}_X)$ ringed space, let $\mathscr{Mod}(X)$ be the category of $\mathfrak{O}_X$-modules

$\mathfrak{I} \in \mathscr{Mod}(X)$ *injective* if $\operatorname{Hom}(-, \mathfrak{I})$ exact;

injective resolution of $\mathfrak{F} \in \mathscr{Mod}(X)$ exact complex

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{I}^0 \longrightarrow \mathfrak{I}^1 \longrightarrow \dots$$

all $\mathfrak{I}^i$, $i \geq 0$ injective; any two injective resolutions are homotopic.

Sheaf Cohomology

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The global section functor $\Gamma(X, -)$ is additive and left exact hence

$$0 \longrightarrow \Gamma(X, \mathfrak{F}) \longrightarrow \Gamma(X, \mathfrak{I}^0) \longrightarrow \Gamma(X, \mathfrak{I}^1) \longrightarrow \dots$$

is exact.

Sheaf Cohomology (finally!)

Definition

Let $(X, \mathfrak{O}_X)$ be a ringed space and $\mathscr{Mod}(X)$ be the category of $\mathfrak{O}_X$-modules. For each $\mathfrak{F} \in \mathscr{Mod}(X)$ the *i-th cohomology group of* $\mathfrak{F}$ is $H^i(X, \mathfrak{F}) = h^i(\Gamma(X, \mathfrak{I}^\bullet))$, $i \geq 0$ where $\mathfrak{I}^\bullet$ is an injective resolution of $\mathfrak{F}$. This yields the cohomology functor

$$H^*(X, -)\colon \mathscr{Mod}(X) \to \mathscr{Ab}.$$

Sheaf Cohomology (finally!)

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Theorem (Grothendieck)

Suppose X is Noetherian with $\dim X = n$. Then $H^i(X, \mathfrak{F}) = 0$ for all $i > n$.

Why did all this work?

because all the categories that appear are abelian:

(i) for any pair of objects, their morphism set has the structure of an abelian group;

(ii) finite products and coproducts exist;

(iii) every morphism has a kernel and a cokernel;

(iv) every monomorphism is the kernel of its cokernel;

(v) every epimorphism is the cokernel of its kernel;

(vi) every morphism is the composition of an epimorphism with a monomorphism.

As a consequence, every bimorphism is an isomorphism.

(And there are no additional analytic structures on the objects making life miserable.)

But for what we want to do

all the categories that appear are non-abelian:

(i) ~~for any pair of objects, their morphism set has the structure of an abelian group;~~

(ii) finite products and coproducts exist;

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(iv) every monomorphism is the kernel of its cokernel;

(v) ~~every epimorphism is the cokernel of its kernel;~~

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(And there *are* additional analytic structures on the objects making life miserable.)

(Commutative) Topology

$\mathscr{C}_1^{*\text{com}}$ category of unital commutative C*-algebras with unital *-homomorphisms as morphisms;

$\mathscr{Comp}$ category of compact Hausdorff spaces with continuous mappings as morphisms.

(Commutative) Topology

$\mathscr{C}_1^{*\text{com}}$ category of unital commutative C^*-algebras with unital $*$-homomorphisms as morphisms;

$\mathscr{C}omp$ category of compact Hausdorff spaces with continuous mappings as morphisms.

$\mathscr{C}omp$		$\mathscr{C}_1^{*\text{com}}$
$X;\quad X \xrightarrow{f} Y$	$\xrightarrow{C}$	$C(X);\quad C(Y) \xrightarrow{C(f)} C(X)$ $C(f)(g) = g \circ f,\ g \in C(Y)$
$\hat{A};\quad \hat{B} \xrightarrow{\hat{\pi}} \hat{A}$ $\hat{\pi}(\varphi) = \varphi \circ \pi,\ \varphi \in \hat{B}$	$\xleftarrow{\ \hat{}\ }$	$A;\quad A \xrightarrow{\pi} B$

Noncommutative Topology

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?

Noncommutative Topology

A C^*-algebra

Noncommutative Topology

A C^*-algebra

$\mathrm{Prim}(A)$ space of primitive ideals of A with hull-kernel topology

$\mathcal{O}_{\mathrm{Prim}(A)} \ni U \quad \rightsquigarrow \quad A(U) = \bigcap_{t \notin U} t;$

$I \trianglelefteq A \quad \rightsquigarrow \quad U(I) = \{t \in \mathrm{Prim}(A) \mid I \nsubseteq t\} \in \mathcal{O}_{\mathrm{Prim}(A)}.$

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hence $V \subseteq U \Rightarrow A(V) \subseteq A(U);$

ringed space $(A, X = \mathrm{Spec}(A), \mathfrak{O}_X)$

"C^*-ringed space" $(A, \mathrm{Prim}(A), \mathfrak{A})$, $\mathfrak{A}$ is a sheaf of C^*-algebras

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Note A separable $\Rightarrow$ closed prime ideals are primitive

P. Ara, M. Mathieu, *Sheaves of C*-algebras*, Math. Nachrichten **283** (2010), 21–39.

Sheaves of C^*-algebras

Example 1. *The multiplier sheaf*

A C^*-algebra with primitive ideal space $\mathrm{Prim}(A)$;

$$\mathfrak{M}_A\colon \mathscr{O}_{\mathrm{Prim}(A)} \to \mathscr{C}_1^*, \quad \mathfrak{M}_A(U) = M(A(U)),$$

where $M(A(U))$ denotes the multiplier algebra of the closed ideal $A(U)$ of A associated to the open subset $U \subseteq \mathrm{Prim}(A)$.
$M(A(U)) \to M(A(V))$, $V \subseteq U$, the restriction homomorphisms.

Proposition

The above functor $\mathfrak{M}_A$ defines a sheaf of C^*-algebras.

Sheaves of C^*-algebras

Example 2. *The injective envelope sheaf*

let $I(B)$ denote the *injective envelope* of B;

$$\mathfrak{I}_A \colon \mathscr{O}_{\mathrm{Prim}(A)} \to \mathscr{C}_1^*, \quad \mathfrak{I}_A(U) = p_U I(A) = I(A(U)),$$

where $p_U = p_{A(U)}$ denotes the unique central open projection in $I(A)$ such that $p_{A(U)} I(A)$ is the injective envelope of $A(U)$.
$I(A(U)) \to I(A(V))$, $V \subseteq U$, given by multiplication by p_V (as $p_V \leq p_U$).

$\{p_U \mid U \in \mathscr{O}_{\mathrm{Prim}(A)}\}$ is a complete Boolean algebra isomorphic to the Boolean algebra of regular open subsets of $\mathrm{Prim}(A)$, and it is precisely the set of projections of the AW^*-algebra $Z(I(A))$.

Noncommutative Topology

Sheaves of operator modules—the analogue of $\mathfrak{O}_X$-modules

X topological space, $\mathfrak{A}$ sheaf of C^*-algebras on X

$\mathfrak{E}\colon \mathscr{O}_X \longrightarrow \mathscr{B}an_1$ (sheaf in $\mathscr{B}an_1$) *right operator $\mathfrak{A}$-module on X*

if, for each $U \in \mathscr{O}_X$, $\mathfrak{E}(U)$ is a (nondegenerate) right operator $\mathfrak{A}(U)$-module and, for $U, V \in \mathscr{O}_X$ with $V \subseteq U$,
$T_{VU}\colon \mathfrak{E}(U) \to \mathfrak{E}(V)$ is completely contractive and

$$T_{VU}(x \cdot a) = T_{VU}(x) \cdot \pi_{VU}(a) \qquad (x \in \mathfrak{E}(U),\ a \in \mathfrak{A}(U)),$$

where $\pi_{VU}\colon \mathfrak{A}(U) \to \mathfrak{A}(V)$ are the restriction maps in $\mathfrak{A}$.

Operator modules

Definition

A right A-module E which at the same time is an operator space is a *right operator A-module* if it satisfies either:

(a) There exist a complete isometry $\Phi\colon E \longrightarrow B(H,K)$, for some Hilbert spaces H, K, and a *-homomorphism $\pi\colon A \longrightarrow B(H)$ such that $\Phi(x \cdot a) = \Phi(x)\pi(a) \quad$ for all $x \in E$, $a \in A$.

(b) The bilinear mapping $E \times A \longrightarrow E$, $(x,a) \mapsto x \cdot a$ extends to a complete contraction $E \otimes_h A \longrightarrow E$.

(c) For each $n \in \mathbb{N}$, $M_n(E)$ is a right Banach $M_n(A)$-module in the canonical way.

E is *nondegenerate* if the linear span of $\{x \cdot a \mid x \in E, a \in A\}$ is dense in E, and it is *unital* if A is unital and $x \cdot 1 = x$ for all $x \in E$.

Noncommutative Topology

Categories we need to consider (for a C^*-algebra A):

$\mathcal{OMod}_A^\infty$ the category with objects the nondegenerate right operator A-modules and morphisms the completely bounded A-module maps;

$\mathcal{OMod}_A^1$ the subcategory of $\mathcal{OMod}_A^\infty$ with morphisms the completely contractive A-module maps.

Noncommutative Topology

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additive, finitely bicomplete

$\mathcal{OMod}_A^1$ the subcategory of $\mathcal{OMod}_A^\infty$ with morphisms the completely contractive A-module maps.

non-additive, bicomplete

None of these categories is abelian!

A close connection between $\mathcal{OMod}_A^1$ and $\mathcal{OMod}_A^\infty$:

Theorem (Ara–Mathieu)

Let A be a C-algebra. In the category $\mathcal{OMod}_A^1$, let Σ denote the class of all bijective morphisms that are invertible in the category $\mathcal{OMod}_A^\infty$. Then Σ is a multiplicative system satisfying the left and the right Ore conditions. Denote the category of fractions with respect to Σ by $\mathcal{OMod}_A^1[\Sigma^{-1}]$, the Gabriel–Zisman localisation. Then the canonical functor*

$$\mathrm{F}\colon \mathcal{OMod}_A^1[\Sigma^{-1}] \longrightarrow \mathcal{OMod}_A^\infty, \quad T/S \longmapsto TS^{-1}$$

is faithful and full, hence an isomorphism.

Homological Algebra

Homological Algebra

in exact categories (à la Quillen)

Homological Algebra

in exact categories (à la Quillen)

An *exact structure* on an additive category $\mathscr{C}$ is a class of kernel–cokernel pairs, closed under isomorphisms, such that the following axioms are satisfied.

[E0] $\forall\ E \in \mathrm{obj}(\mathscr{C})$: 1_E is an admissible monomorphism;

[E0$^{\mathrm{op}}$] $\forall\ E \in \mathrm{obj}(\mathscr{C})$: 1_E is an admissible epimorphism;

[E1] the class of admissible monomorphisms is closed under composition;

[E1$^{\mathrm{op}}$] the class of admissible epimorphisms is closed under composition;

[E2] the push-out of an admissible monomorphism along an arbitrary morphism exists and yields an admissible monomorphism;

[E2$^{\mathrm{op}}$] the pull-back of an admissible epimorphism along an arbitrary morphism exists and yields an admissible epimorphism.

Th. Bühler, *Exact categories*, Expo. Math. **28** (2010), 1–69.

Homological Algebra

in exact categories (à la Quillen)

Definition

Let A be a C^*-algebra. We endow the additive category $\mathcal{OMod}_A^\infty$ with the exact structure of all kernel–cokernel pairs

$$E_1 \overset{M}{\rightarrowtail} E_2 \overset{P}{\twoheadrightarrow} E_3$$

where $E_i \in \mathrm{obj}(\mathcal{OMod}_A^\infty)$, $1 \leq i \leq 3$, M is a monomorphism in $\mathcal{OMod}_A^\infty$ with closed range and completely bounded inverse, P is a completely open mapping in $\mathcal{OMod}_A^\infty$ (in particular, surjective) and $\ker P = \operatorname{im} M$.

Homological Algebra

in exact categories (à la Quillen)

Proposition

Let A be a C-algebra. The class of all kernel–cokernel pairs in $\mathscr{OMod}_A^\infty$ is an exact structure on $\mathscr{OMod}_A^\infty$.*

Homological Algebra—exact sequences in $\mathcal{OMod}_A^1$:

each kernel-cokernel pair in $\mathcal{OMod}_A^\infty$ is isomorphic to one in which the admissible monomorphism is completely isometric and the admissible epimorphism is a complete quotient mapping:

$$\begin{array}{ccccc} E_1 & \overset{M}{\rightarrowtail} & E_2 & \overset{P}{\twoheadrightarrow} & E_3 \\ \uparrow \cong & & \| & & \cong \uparrow \\ \ker P & \rightarrowtail & E_2 & \twoheadrightarrow & E_2/\operatorname{im} M \end{array}$$

Definition

Let $E_i \in \text{obj}(\mathcal{OMod}_A^1)$, $1 \leq i \leq 3$. We call a sequence

$$0 \longrightarrow E_1 \xrightarrow{M} E_2 \xrightarrow{P} E_3 \longrightarrow 0$$

exact in $\mathcal{OMod}_A^1$ if M is a complete isometry, P is a complete quotient map and $\operatorname{im} M = \ker P$.

Homological Algebra—exact sequences in $\mathcal{OMod}_A^1$:

one of the problems. . .

the sequences

$$0 \longrightarrow E \longrightarrow E \oplus F \longrightarrow F \longrightarrow 0$$

and

$$0 \longrightarrow E \longrightarrow E \times F \longrightarrow F \longrightarrow 0$$

are non-isomorphic split exact sequences in $\mathcal{OMod}_A^1$ though they yield the same kernel–cokernel pair in $\mathcal{OMod}_A^\infty$.

Injective objects

Proposition

A right operator A-module E is injective if and only if $CB_A(-, E)$ is exact and, for each admissible monomorphism M in $\mathcal{OMod}_A^1$, M^* maps the unit ball of its domain onto the unit ball of its codomain.

Glimpse into the proof

In order to verify axiom [E2] let $E \overset{M}{\rightarrowtail} F$ be an admissible monomorphism in $\mathscr{OMod}_A^\infty$ and take any $S \in CB_A(E, G)$. Let $H = \{(Mx, -Sx) \mid x \in E\}$ which is a closed submodule of $F \oplus G$: by hypothesis there is $N\colon \operatorname{im} M \to E$ which is a completely bounded A-module map such that $NM = \mathrm{id}_E$. Thus, if $Mx_n \to y$ for some sequence $(x_n)_{n\in\mathbb{N}}$ in E, $y \in \operatorname{im} M$ and $x_n = NMx_n \to Ny$. Consequently, $(Mx_n, -Sx_n) \to (MNy, -SNy) \in H$. Denote by $F \oplus_H G$ the quotient operator A-module $F \oplus_H G = \big(F \oplus G\big)/H$.

Glimpse into the proof

In the pushout diagram

$$\begin{array}{ccc} E & \overset{M}{\rightarrowtail} & F \\ {\scriptstyle S}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ G & \underset{M'}{\rightarrowtail} & F \oplus_H G \end{array}$$

the mappings $T\colon y \mapsto (y, 0) + H$ and $M'\colon z \mapsto (0, z) + H$ are complete contractions and $TM = M'S$. By construction, M' is injective: $M'z = 0$ implies that $(0, z) = (Mx, -Sx)$ for some $x \in E$ whence $x = 0$ as M is injective and so $z = 0$. Unless $S = 0$ (in which case $M' = M$) we introduce an equivalent matrix norm structure on G by $|||z|||_n = \gamma \, \|z\|_n$, $z \in M_n(G)$, $n \in \mathbb{N}$, where $\gamma = (\|N\|_{cb} \, \|S\|_{cb})^{-1}$. As the class of admissible monomorphisms in stable under isomorphisms, it suffices to consider M' with respect to this equivalent operator module structure.

Glimpse into the proof

For each $x \in M_n(E)$ we have

$$|||S_n x|||_n = \gamma \, \|S_n x\|_n \leq \gamma \, \|S\|_{cb} \|x\|_n = \frac{1}{\|N\|_{cb}} \|x\|_n \leq \|M_n x\|_n,$$

where $S_n \colon M_n(E) \to M_n(G)$ and $M_n \colon M_n(E) \to M_n(F)$ denote the ampliations in the above diagram decorated with $n \times n$-matrices throughout.

Note that the operator space structure on $F \oplus_H G$ is given by

$$M_n(F \oplus_H G) = \big(M_n(F) \oplus M_n(G)\big)/M_n(H), \;\; n \in \mathbb{N}.$$

In particular, for $z \in M_n(G)$,

$$\|M_n' z\|_n = \inf_{x \in M_n(E)} \|(0, z) + (M_n x, -S_n x)\|_n.$$

Glimpse into the proof

Thus, for each $n \in \mathbb{N}$, there exists $x \in M_n(E)$ such that

$$\begin{aligned} \|M'_n z\|_n &\geq \frac{1}{2}\,\|(0,z) + (M_n x, -S_n x)\|_n = \frac{1}{2}\,\|(M_n x, z - S_n x)\|_n \\ &= \frac{1}{2}\,(\|M_n x\|_n + |||z - S_n x|||_n) \\ &\geq \frac{1}{2}\,(|||S_n x|||_n + |||z - S_n x|||_n) \\ &\geq \frac{1}{2}\,|||z|||_n = \frac{\gamma}{2}\,\|z\|_n. \end{aligned}$$

As a result, M' has a completely bounded inverse with cb-norm at most $\frac{2}{\gamma}$ and hence is an admissible monomorphism in $\mathcal{OMod}^{\infty}_{A}$:

Glimpse into the proof

if $M = \operatorname{Coker} P$

$$\begin{array}{ccccc} E & \overset{M}{\rightarrowtail} & F & \overset{P}{\twoheadrightarrow} & K \\ {\scriptstyle S}\downarrow & & \vdots\,{\scriptstyle T} & & \| \\ G & \underset{M'}{\dashrightarrow} & F \oplus_H G & \underset{P'}{\longrightarrow} & K \end{array}$$

then $P' \colon (y, z) + H \mapsto Py$ is a well defined epimorphism and $M' = \operatorname{Coker} P'$. □

what is next?

It becomes apparent that, in order to move any further, we need to understand sheaves in various categories much better; this will be the topic of the next talk.