Local Multipliers and Derivations, Sheaves of C-Algebras and Cohomology*

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Part IV: Sheaves in general categories

Sheaves in categories

X topological space;
$\mathcal{O}_X$ category of open subsets (with open subsets U as objects and $V \to U$ if and only if $V \subseteq U$).

$\mathcal{C}$ bicomplete category with equalisers and coequalisers.

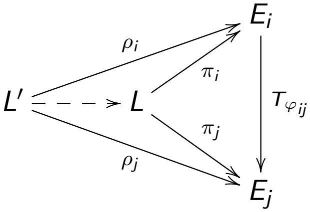
Definition

A *presheaf on X in $\mathcal{C}$* is a contravariant functor $\mathfrak{F}\colon \mathcal{O}_X \to \mathcal{C}$.

A *sheaf with values in $\mathcal{C}$* is a presheaf $\mathfrak{F}$ such that $\mathfrak{F}(\emptyset) = 0$ and, for every open subset U of X and every open cover $U = \bigcup_i U_i$, the maps $\mathfrak{F}(U) \to \mathfrak{F}(U_i)$ are the limit of the diagrams $\mathfrak{F}(U_i) \to \mathfrak{F}(U_i \cap U_j)$ for all i, j.

Limits in categories

Let $\mathscr{I} \longrightarrow \mathscr{C}$ be a small diagram; we write $E_i \in \mathscr{C}$ for the image of an object $i \in \mathscr{I}$ and, if $\varphi_{ij} \colon i \to j$ is a morphism, we denote its image by $T_{\varphi_{ij}} \colon E_i \to E_j$. An object $L \in \mathscr{C}$ together with morphisms $\pi_i \colon L \to E_i$, $i \in \mathscr{I}$ is *a limit* of the diagram if they make the diagram below commutative and is final with this property.



Also called an *inverse limits* or *projective limit*.

Sheaves in categories

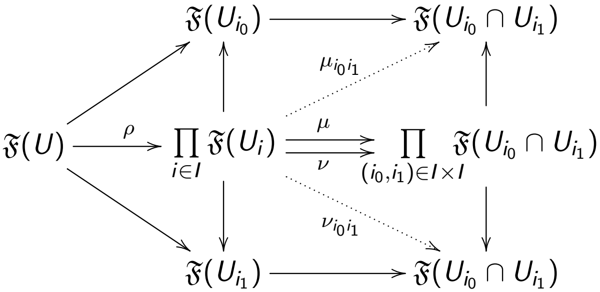
Let $U \in \mathcal{O}_X$. Let I be a set and $U = \bigcup_{i \in I} U_i$ with $U_i \in \mathcal{O}_X$. Let the index category $I^{(2)}$ consist of unordered pairs of elements in I (we allow the case of singleton sets in $I^{(2)}$)
where $\{i, j\} \to \{k, \ell\}$ if $\{i, j\} \subseteq \{k, \ell\}$. Consider the diagram

$$I^{(2)} \to \{U_{ij} = U_i \cap U_j \mid \{i, j\} \in I^{(2)}\} \subseteq \mathcal{O}_U, \quad \{i, j\} \mapsto U_{ij}$$

composed with the functor $\mathfrak{F} \colon \mathcal{O}_X \to \mathscr{C}$. Then $\mathfrak{F}(U)$ is the limit of the diagram

$$\begin{array}{ccccc} & & \mathfrak{F}(U_i) & & \\ & \nearrow & & \searrow & \\ \mathfrak{F}(U) & & & & \mathfrak{F}(U_i \cap U_j) \\ & \searrow & & \nearrow & \\ & & \mathfrak{F}(U_j) & & \end{array}$$

Sheaves in categories



The sheaf property is the requirement that the morphism ρ is the equaliser of the pair (μ, ν).

Sheaves in *concrete* categories

Notation and Terminology:

suppose $\mathscr{C}$ is a concrete category

the elements of $\mathfrak{F}(U)$ are called *sections* over $U \in \mathcal{O}_X$;

by $s_{|V}$, $V \subseteq U$ open, we mean the "restriction" of $s \in \mathfrak{F}(U)$ to V; i.e., the image of s in $\mathfrak{F}(V)$ under $\rho_{VU} \colon \mathfrak{F}(U) \to \mathfrak{F}(V)$;

the unique gluing property of a sheaf can be expressed as follows:

for each compatible family of sections $s_i \in \mathfrak{F}(U_i)$, i.e., $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for all i, j, there is a unique section $s \in \mathfrak{F}(U)$ such that $s_{|U_i} = s_i$ for all i.

Sheaves in categories

$\mathfrak{F}$, $\mathfrak{G}$ (pre)sheaves on X

morphism $\varphi\colon \mathfrak{F} \to \mathfrak{G}$ is a natural transformation, i.e.,

$$\begin{array}{ccc} \mathfrak{F}(U) & \xrightarrow{\varphi_U} & \mathfrak{G}(U) \\ {\scriptstyle \rho_{VU}}\downarrow & & \downarrow{\scriptstyle \rho'_{VU}} \\ \mathfrak{F}(V) & \xrightarrow[\varphi_V]{} & \mathfrak{G}(V) \end{array}$$

is commutative for all $U, V \in \mathscr{O}_X$, $V \subseteq U$;

hence we have the categories of sheaves on X, $\mathscr{S}h(X, \mathscr{C})$, and of presheaves, $\mathscr{P}\mathscr{S}h(X, \mathscr{C})$.

Sheaves in categories

Stalks

$\mathfrak{F}$ presheaf on X, $t \in X$;

stalk of $\mathfrak{F}$ at t is defined as $\mathsf{F}_t = \varinjlim_{\mathscr{U}_t} \mathfrak{F}(U)$,

where $\mathscr{U}_t$ denotes the downward directed family of open neighbourhoods of t and $\varinjlim$ denotes the (directed) colimit in $\mathscr{C}$.

With stalks we can build bundles and with bundles we can get sheaves again.

Sheaves in categories

An Example

Let $\mathfrak{F}$ be a presheaf on X. For each $U \in \mathscr{O}_X$, put $\mathfrak{F}^p(U) = \prod_{t \in U} \mathsf{F}_t$ and, for $V \subseteq U$, set ρ_{VU} the canonical morphism from $\prod_{t \in U} \mathsf{F}_t \to \prod_{t \in V} \mathsf{F}_t$.
In this way we obtain the *product sheaf associated with* $\mathfrak{F}$.

We shall assume that the canonical morphism $\sigma_U \colon \mathfrak{F}(U) \to \mathfrak{F}^p(U)$ is a monomorphism whenever $\mathfrak{F}$ is a sheaf.

Sheaves in categories

The stalk functor

Suppose $\mathfrak{F}$ and $\mathfrak{G}$ are presheaves on X and $\varphi\colon \mathfrak{F} \to \mathfrak{G}$ is a morphism of presheaves.
For each $t \in X$ there is a unique morphism $\varphi_t\colon \mathsf{F}_t \to \mathsf{G}_t$.
In this way, we obtain the *stalk functor at* t: $\mathscr{P}\mathcal{S}h(X, \mathscr{C}) \to \mathscr{C}$.

Sheaves in categories

The stalk functor

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Properties:

- Let $\varphi^{(1)}, \varphi^{(2)}\colon \mathfrak{F} \to \mathfrak{G}$ be morphisms of sheaves. Then $\varphi^{(1)} = \varphi^{(2)}$ if and only if $\varphi_t^{(1)} = \varphi_t^{(2)}$ for all $t \in X$.

Sheaves in categories

The stalk functor

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Properties:

- Let $\varphi^{(1)}, \varphi^{(2)}\colon \mathfrak{F} \to \mathfrak{G}$ be morphisms of sheaves. Then $\varphi^{(1)} = \varphi^{(2)}$ if and only if $\varphi_t^{(1)} = \varphi_t^{(2)}$ for all $t \in X$.
- Let $\varphi\colon \mathfrak{F} \to \mathfrak{G}$ be a morphism of sheaves. Then
  (i) φ is a monomorphism if φ_t is a monomorphism for all $t \in X$;
  (ii) φ is an epimorphism if φ_t is an epimorphism for all $t \in X$;
  (iii) φ is an isomorphism only if φ_t is an isomorphism for all $t \in X$.

Various operations with sheaves

Restriction of a sheaf. Let $\mathfrak{F} \in \mathcal{S}h(X, \mathscr{C})$. For every open subset $U \in \mathscr{O}_X$, the restriction $\mathfrak{F}_{|U}$ to U is a sheaf on U.

The *support* $\operatorname{supp} \mathfrak{F}$ of a sheaf $\mathfrak{F}$ is the complement of the union of all $U \in \mathscr{O}_X$ such that $\mathfrak{F}_{|U} = 0$ (i.e., is the constant zero sheaf). Thus $\mathfrak{F}_{|X \setminus \operatorname{supp} \mathfrak{F}} = 0$.

Sub(pre)sheaf. A presheaf $\mathfrak{F}$ is called a *subpresheaf* of a presheaf $\mathfrak{G}$ if $\mathfrak{F}(U)$ is a subobject of $\mathfrak{G}(U)$ for all $U \in \mathscr{O}_X$ such that the restriction morphisms of $\mathfrak{G}$ induce the restriction morphisms of $\mathfrak{F}$. If $\mathfrak{F}$ and $\mathfrak{G}$ are sheaves, then $\mathfrak{F}$ is called a *subsheaf* of $\mathfrak{G}$.
Notation $\quad \mathfrak{F} \subseteq \mathfrak{G}$.

Various operations with sheaves

Sheafification of a presheaf.

The aim is to define a functor from $\mathscr{PS}h(X,\mathscr{C})$ to $\mathscr{S}h(X,\mathscr{C})$ that acts as a left adjoint to the forgetful functor from $\mathscr{S}h(X,\mathscr{C})$ to $\mathscr{PS}h(X,\mathscr{C})$.

Various operations with sheaves

Sheafification of a presheaf.

Basic assumptions:

A concrete category $\mathscr{C}$ whose objects carry some analytic structure together with its algebraic 'counterpart', denoted by $\mathscr{C}_{\mathrm{a}}$.

e.g., $\mathscr{B}an_1$, $\mathscr{C}^*$ or $\mathscr{OM}od_A^1$ (Banach spaces, C^*-algebras or operator modules over a C^*-algebra A)

together with $\mathscr{V}ec_{\mathbb{C}}$, $\mathscr{A}lg_{\mathbb{C}}$ or $\mathscr{M}od_A$ (complex vector spaces, complex algebras or right modules (over the C^*-algebra A))

together with the forgetful functors $\mathscr{PS}h(X, \mathscr{C}) \to \mathscr{PS}h(X, \mathscr{C}_{\mathrm{a}})$ and $\mathscr{S}h(X, \mathscr{C}) \to \mathscr{S}h(X, \mathscr{C}_{\mathrm{a}})$.

$\mathscr{C}_{\mathrm{a}}$ is *balanced*, i.e., bimorphisms are isomorphisms; in our cases, the isomorphisms in $\mathscr{C}_{\mathrm{a}}$ will be the bijective morphisms.

Various operations with sheaves

Sheafification of a presheaf.

Main steps:

Various operations with sheaves

Sheafification of a presheaf.

Main steps: For each $U \in \mathcal{O}_X$, let

$$\mathfrak{F}^0(U) = \{x \in \mathfrak{F}^p(U) \mid \forall\, t \in U\, \exists\, W_t \in \mathcal{U}_t,\ W_t \subseteq U \text{ and } y \in \mathfrak{F}(W_t) \text{ such that } x_s = y(s) \text{ for all } s \in W_t\}.$$

Various operations with sheaves

Sheafification of a presheaf.

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For each $U, V \in \mathcal{O}_X$, $V \subseteq U$, we have a commutative diagram in $\mathscr{C}_\mathrm{a}$

$$\begin{array}{ccccc} \mathfrak{F}(U) & \longrightarrow & \mathfrak{F}^0(U) & \longrightarrow & \mathfrak{F}^p(U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{F}(V) & \longrightarrow & \mathfrak{F}^0(V) & \longrightarrow & \mathfrak{F}^p(V) \end{array}$$

Thus there is a canonical morphism of presheaves $\sigma^0 \colon \mathfrak{F} \to \mathfrak{F}^0$ in $\mathscr{C}_\mathrm{a}$ and $\mathfrak{F}^0$ defines a subsheaf of $\mathfrak{F}^p$ in $\mathscr{C}_\mathrm{a}$.

Various operations with sheaves

Sheafification of a presheaf.

Main steps: Contrary to the purely algebraic situation:

the stalks F_t of $\mathfrak{F}$ and F_t^0 of $\mathfrak{F}^0$ may not be isomorphic!

Various operations with sheaves

Sheafification of a presheaf.

Main steps:

Let $\mathfrak{F} \in \mathscr{PS}h(X, \mathscr{C})$. For $U \in \mathscr{O}_X$, put $\widetilde{\mathfrak{F}}(U) = \bigcap_{\mathfrak{G} \in \mathsf{S}_{\mathfrak{F}}} \mathfrak{G}(U)$, where $\mathsf{S}_{\mathfrak{F}}$ consists of all subsheaves $\mathfrak{G}$ of $\mathfrak{F}^p$ in $\mathscr{C}$ containing $\mathfrak{F}^0$ as a subsheaf in $\mathscr{C}_{\mathsf{a}}$.

Various operations with sheaves

Sheafification of a presheaf.

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For all $\mathfrak{G} \in \mathsf{S}_{\mathfrak{F}}$, the restriction morphisms $\mathfrak{G}(U) \to \mathfrak{G}(V)$, $U, V \in \mathscr{O}_X$, $V \subseteq U$ are the ones induced by $\mathfrak{F}^p(U) \to \mathfrak{F}^p(V)$; thus $\widetilde{\mathfrak{F}}$ is a subsheaf of $\mathfrak{F}^p$.

Various operations with sheaves

Sheafification of a presheaf.

Main steps:

Let $\mathfrak{F} \in \mathscr{P}\mathscr{S}h(X, \mathscr{C})$. For $U \in \mathscr{O}_X$, put $\widetilde{\mathfrak{F}}(U) = \bigcap_{\mathfrak{G} \in \mathrm{S}_{\mathfrak{F}}} \mathfrak{G}(U)$, where $\mathrm{S}_{\mathfrak{F}}$ consists of all subsheaves $\mathfrak{G}$ of $\mathfrak{F}^p$ in $\mathscr{C}$ containing $\mathfrak{F}^0$ as a subsheaf in $\mathscr{C}_{\mathrm{a}}$.

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Define a functor $\mathscr{P}\mathscr{S}h(X, \mathscr{C}) \to \mathscr{S}h(X, \mathscr{C})$, $\eta \colon \mathfrak{F} \mapsto \widetilde{\mathfrak{F}}$ as follows: for a morphism $\varphi \colon \mathfrak{F}^1 \to \mathfrak{F}^2$ of presheaves in $\mathscr{C}$, put $\tilde{\varphi}_U$ the restriction to $\widetilde{\mathfrak{F}^1}(U)$ of the extension of φ_U to $(\mathfrak{F}^1)^p(U)$ in order to obtain a morphism $\tilde{\varphi} \colon \widetilde{\mathfrak{F}^1} \to \widetilde{\mathfrak{F}^2}$.

Various operations with sheaves

Sheafification of a presheaf.

Theorem

Under the assumption that $\widetilde{\mathsf{F}}_t = \mathsf{F}_t$ *for all* $t \in X$*, the above functor is a sheafification functor; that is, for every morphism of presheaves* $\varphi\colon \mathfrak{F} \to \mathfrak{G}$ *from a presheaf* $\mathfrak{F} \in \mathscr{P}\mathscr{S}h(X, \mathscr{C})$ *into a sheaf* $\mathfrak{G} \in \mathscr{S}h(X, \mathscr{C})$ *there is a unique morphism* $\tilde{\varphi}\colon \widetilde{\mathfrak{F}} \to \mathfrak{G}$ *making the diagram below commutative for each* $U \in \mathscr{O}_X$

$$\begin{array}{ccc} \mathfrak{F}(U) & \xrightarrow{\eta_U} & \widetilde{\mathfrak{F}}(U) \\ & \searrow^{\varphi_U} & \downarrow{\scriptstyle \tilde{\varphi}_U} \\ & & \mathfrak{G}(U) \end{array}$$

Various operations with sheaves

Sheafification of a presheaf.

Theorem

In our analytic categories to be considered, the sheafification functor $\eta\colon \mathfrak{F} \mapsto \widetilde{\mathfrak{F}}$ *exists and is a left adjoint of the forgetful functor* $\mathcal{S}h(X,\mathscr{C}) \to \mathscr{P}\mathcal{S}h(X,\mathscr{C})$, $\mathfrak{G} \mapsto \underline{\mathfrak{G}}$. *That is,*

$$\mathrm{Mor}_{\mathcal{S}h(X,\mathscr{C})}(\widetilde{\mathfrak{F}}, \mathfrak{G}) \cong \mathrm{Mor}_{\mathscr{P}\mathcal{S}h(X,\mathscr{C})}(\mathfrak{F}, \underline{\mathfrak{G}}).$$

Various operations with sheaves

Sheafification of a presheaf.

Theorem

In our analytic categories to be considered, the sheafification functor $\eta\colon \mathfrak{F} \mapsto \widetilde{\mathfrak{F}}$ *exists and is a left adjoint of the forgetful functor* $\mathcal{S}h(X,\mathscr{C}) \to \mathscr{P}\mathcal{S}h(X,\mathscr{C})$, $\mathfrak{G} \mapsto \underline{\mathfrak{G}}$. *That is,*

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the sheafification of a presheaf is needed for various other constructions with sheaves

Various operations with sheaves

Product of sheaves.

Let $\mathfrak{E}$, $\mathfrak{F} \in \mathscr{PS}h(X, \mathscr{C})$ and let $\{\rho_{VU}\}$, $\{\sigma_{VU}\}$, $V \subseteq U$ denote the restriction morphisms in $\mathfrak{E}$ and $\mathfrak{F}$, respectively. For each $U \in \mathscr{O}_X$, we let $(\mathfrak{E} \times \mathfrak{F})(U) = \mathfrak{E}(U) \times \mathfrak{F}(U)$ and for $V \in \mathscr{O}_X$, $V \subseteq U$, let $(\rho \times \sigma)_{VU} = \rho_{VU} \times \sigma_{VU}$. With these restriction morphisms, $\mathfrak{E} \times \mathfrak{F} \in \mathscr{PS}h(X, \mathscr{C})$ and it is the product in $\mathscr{PS}h(X, \mathscr{C})$ with the canonical product morphisms $\mathfrak{E} \times \mathfrak{F} \to \mathfrak{E}$ and $\mathfrak{E} \times \mathfrak{F} \to \mathfrak{F}$.

In the case that $\mathfrak{E}$ and $\mathfrak{F}$ are sheaves, the product $\mathfrak{E} \times \mathfrak{F}$ is a sheaf as well.

Various operations with sheaves

Pullback of sheaves.

It follows from general principles that pullbacks exist in $\mathcal{S}h(X, \mathscr{C})$. Let $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G} \in \mathscr{P}\mathcal{S}h(X, \mathscr{C})$ and let $\rho\colon \mathfrak{E} \to \mathfrak{G}$ and $\tau\colon \mathfrak{F} \to \mathfrak{G}$ be given. The pullback diagram

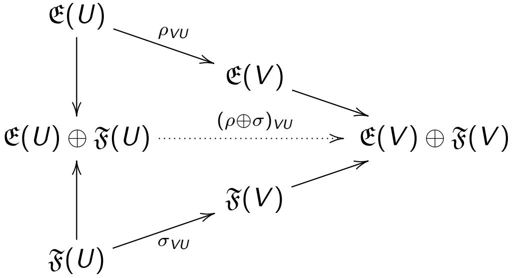
$$\begin{array}{ccc} \mathfrak{E} \times_{\mathfrak{G}} \mathfrak{F} & \xrightarrow{\ \overline{\rho}\ } & \mathfrak{F} \\ {\scriptstyle \overline{\tau}}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathfrak{E} & \xrightarrow[\ \rho\ ]{} & \mathfrak{G} \end{array}$$

is defined by $(\mathfrak{E} \times_{\mathfrak{G}} \mathfrak{F})(U) = \mathfrak{E}(U) \times_{\mathfrak{G}(U)} \mathfrak{F}(U)$, $U \in \mathscr{O}_X$ with the restriction morphisms induced by those of $\mathfrak{E} \times \mathfrak{F}$ and $\overline{\rho}$, $\overline{\tau}$ induced by the canonical projection morphisms. It is easy to check that $\mathfrak{E} \times_{\mathfrak{G}} \mathfrak{F}$ is a sheaf provided $\mathfrak{E}$, $\mathfrak{F}$ and $\mathfrak{G}$ are sheaves on X.

Various operations with sheaves

Coproduct of sheaves.

Let $\mathfrak{E}$, $\mathfrak{F} \in \mathscr{PS}h(X, \mathscr{C})$ and let $\{\rho_{VU}\}$, $\{\sigma_{VU}\}$, $V \subseteq U$ denote the restriction morphisms in $\mathfrak{E}$ and $\mathfrak{F}$, respectively. For each $U \in \mathscr{O}_X$, we let $(\mathfrak{E} \underline{\oplus} \mathfrak{F})(U) = \mathfrak{E}(U) \oplus \mathfrak{F}(U)$, where "$\oplus$" denotes the coproduct in $\mathscr{C}$. Suppose $V \in \mathscr{O}_X$, $V \subseteq U$.



It is easy to check that in this way a presheaf $\mathfrak{E} \underline{\oplus} \mathfrak{F}$ on X is defined which is the coproduct in $\mathscr{P}\mathscr{S}h(X, \mathscr{C})$.

Let $\mathfrak{E}$, $\mathfrak{F} \in \mathscr{S}h(X, \mathscr{C})$. As $\mathfrak{E} \underline{\oplus} \mathfrak{F}$ may not be a sheaf we define the *sheaf coproduct* $\mathfrak{E} \oplus \mathfrak{F}$ of $\mathfrak{E}$ and $\mathfrak{F}$ as the sheafification of $\mathfrak{E} \underline{\oplus} \mathfrak{F}$.

This will be the coproduct in the category $\mathscr{S}h(X, \mathscr{C})$.

Various operations with sheaves

Pushout of sheaves.

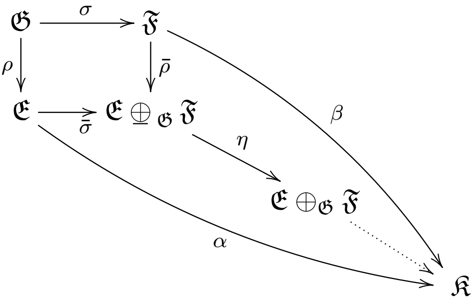
It follows from general principles that pushouts exist in $\mathcal{S}h(X, \mathscr{C})$. Let $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G} \in \mathscr{P}\mathcal{S}h(X, \mathscr{C})$ and let $\rho\colon \mathfrak{G} \to \mathfrak{E}$ and $\sigma\colon \mathfrak{G} \to \mathfrak{F}$ be given. The pushout diagram

$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{\ \sigma\ } & \mathfrak{F} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\bar\rho} \\ \mathfrak{E} & \xrightarrow[\ \bar\sigma\ ]{} & \mathfrak{E} \oplus_{\mathfrak{G}} \mathfrak{F} \end{array}$$

is defined as follows. Let $\iota_{\mathfrak{E}}$ and $\iota_{\mathfrak{F}}$ be the canonical injections from $\mathfrak{E}$ and $\mathfrak{F}$ to $\mathfrak{E} \oplus \mathfrak{F}$, respectively. Then $\mathfrak{E} \oplus_{\mathfrak{G}} \mathfrak{F}$ is the sheafification of the coequaliser (in $\mathscr{P}\mathcal{S}h(X, \mathscr{C})$) of $(\iota_{\mathfrak{E}} \circ \rho, \iota_{\mathfrak{F}} \circ \sigma)$. Together with the unique induced morphisms $\bar\sigma$ and $\bar\rho$, this sheaf is the pushout in $\mathcal{S}h(X, \mathscr{C})$ provided $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ are sheaves.

Various operations with sheaves

Pushout of sheaves.



Various operations with sheaves

Direct image sheaf.

Let $f\colon X \to Y$ be a continuous mapping. Let $\mathfrak{F} \in \mathscr{PS}h(X, \mathscr{C})$. Let $f^{-1}\colon \mathscr{O}_Y \to \mathscr{O}_X$ be the (covariant) functor $V \mapsto f^{-1}(V)$. Putting $f_*\mathfrak{F} = \mathfrak{F} \circ f^{-1}$ yields a functor $f_*\colon \mathscr{PS}h(X, \mathscr{C}) \to \mathscr{PS}h(Y, \mathscr{C})$ which sends sheaves on X to sheaves on Y since, if $V = \bigcup_i V_i$ is an open covering of $V \in \mathscr{O}_Y$, then $f^{-1}(V) = \bigcup_i f^{-1}(V_i)$ is an open covering in $\mathscr{O}_X$.

Evidently, if $g\colon Y \to Z$ is another continuous mapping, then $(g \circ f)_* = g_* \circ f_*$.

Various operations with sheaves

Inverse image sheaf.

Let $\mathfrak{G} \in \mathcal{S}h(Y, \mathscr{C})$ and let $f\colon X \to Y$ be a continuous mapping.
Step 1 in the construction of the *inverse image* $f^{-1}\mathfrak{G}$:

For $U \in \mathscr{O}_X$, let $\mathscr{U}_{f(U)}$ denote the full subcategory of $\mathscr{O}_Y$ whose objects are the open subsets $V \in \mathscr{O}_Y$ that contain $f(U)$.
If $U_1, U_2 \in \mathscr{O}_X$ satisfy $U_1 \subseteq U_2$ then $f(U_1) \subseteq f(U_2)$ and hence $\mathscr{U}_{f(U_2)}$ is a full subcategory of $\mathscr{U}_{f(U_1)}$.

Various operations with sheaves

Inverse image sheaf.

Let $\mathfrak{G} \in \mathcal{S}h(Y, \mathscr{C})$ and let $f\colon X \to Y$ be a continuous mapping.
Step 1 in the construction of the *inverse image* $f^{-1}\mathfrak{G}$:

Define $f_p\mathfrak{G}$ by $f_p\mathfrak{G}(U) = \operatorname{colim}_{\mathscr{U}_{f(U)}} \mathfrak{G}(V)$, where $W \in \mathscr{U}_{f(U_2)}$:

$$
\begin{array}{ccccc}
 & \nearrow & \operatorname{colim}_{\mathscr{U}_{f(U_2)}} \mathfrak{G}(V) & = & f_p\mathfrak{G}(U_2) \\
\mathfrak{G}(W) & & \downarrow & & \downarrow \rho_{U_1 U_2} \\
 & \searrow & \operatorname{colim}_{\mathscr{U}_{f(U_1)}} \mathfrak{G}(V) & = & f_p\mathfrak{G}(U_1)
\end{array}
$$

In this way we obtain a functor $f_p\colon \mathcal{S}h(Y, \mathscr{C}) \to \mathscr{P}\mathcal{S}h(X, \mathscr{C})$, the *pullback presheaf functor*.

Various operations with sheaves

Inverse image sheaf.

Let $\mathfrak{G} \in \mathscr{S}h(Y, \mathscr{C})$ and let $f\colon X \to Y$ be a continuous mapping.
Step 2 in the construction of the *inverse image* $f^{-1}\mathfrak{G}$:
Define $f^{-1}\mathfrak{G} = \widetilde{f_p\mathfrak{G}}$, the sheafification of $f_p\mathfrak{G}$.

Various operations with sheaves

Inverse image sheaf.

Let $\mathfrak{G} \in \mathcal{S}h(Y, \mathscr{C})$ and let $f\colon X \to Y$ be a continuous mapping.
Step 2 in the construction of the *inverse image* $f^{-1}\mathfrak{G}$:

It turns out that f^{-1} is a left adjoint to the direct image functor f_*.

Proposition

Let $f\colon X \to Y$ be a continuous mapping between the topological spaces X and Y. Let $\mathfrak{F} \in \mathcal{S}h(X, \mathscr{C})$ and $\mathfrak{G} \in \mathcal{S}h(Y, \mathscr{C})$. Then

$$\mathrm{Mor}_{\mathcal{S}h(X,\mathscr{C})}(f^{-1}\mathfrak{G}, \mathfrak{F}) \cong \mathrm{Mor}_{\mathcal{S}h(Y,\mathscr{C})}(\mathfrak{G}, f_*\mathfrak{F}).$$

Moreover there is a canonical isomorphism of stalks $(\mathrm{f}^{-1}\mathrm{G})_t \cong \mathrm{G}_{f(t)}$ for every $t \in X$.

The categories we are interested in

An Example

Let A be a C^*-algebra, and let X be a topological space (e.g., $X = \mathrm{Prim}(A)$ could be its primitive ideal space).
Suppose $E \in \mathcal{OMod}_A^1$ is a nondegenerate right operator A-module.
Then E is a unital right operator $M(A)$-module via the module action $x \cdot m = \lim_\lambda x \cdot (e_\lambda m)$ for $x \in E$, $m \in M(A)$ and (e_λ) an approximate identity of A.

Suppose $\mathfrak{E}$ is a sheaf in $\mathcal{OMod}_A^1$ over X. By the foregoing, $\mathfrak{E}(U)$ is a unital $M(A)$-module. For $V \subseteq U$, the connecting map $T_{VU}\colon \mathfrak{E}(U) \to \mathfrak{E}(V)$ yields a morphism in $\mathcal{OMod}_{M(A)}^1$.
In this way, we obtain an associated sheaf in $\mathcal{OMod}_{M(A)}^1$ consisting of unital modules.

Recall: operator modules

Definition

A right A-module E which at the same time is an operator space is a *right operator A-module* if it satisfies either:

(a) There exist a complete isometry $\Phi\colon E \longrightarrow B(H, K)$, for some Hilbert spaces H, K, and a *-homomorphism $\pi\colon A \longrightarrow B(H)$ such that $\Phi(x \cdot a) = \Phi(x)\pi(a) \quad$ for all $x \in E$, $a \in A$.

(b) The bilinear mapping $E \times A \longrightarrow E$, $(x, a) \mapsto x \cdot a$ extends to a complete contraction $E \otimes_h A \longrightarrow E$.

(c) For each $n \in \mathbb{N}$, $M_n(E)$ is a right Banach $M_n(A)$-module in the canonical way.

E is *nondegenerate* if the linear span of $\{x \cdot a \mid x \in E, a \in A\}$ is dense in E, and it is *unital* if A is unital and $x \cdot 1 = x$ for all $x \in E$.

"C^*-ringed spaces"

Sheaves of operator modules over sheaves of C^*-algebras

X topological space, $\mathfrak{A}$ sheaf of C^*-algebras on X

$\mathfrak{E}\colon \mathscr{O}_X \longrightarrow \mathscr{B}an_1$ (sheaf in $\mathscr{B}an_1$) *right operator $\mathfrak{A}$-module on X*

if, for each $U \in \mathscr{O}_X$, $\mathfrak{E}(U)$ is a (nondegenerate) right operator $\mathfrak{A}(U)$-module and, for $U, V \in \mathscr{O}_X$ with $V \subseteq U$,
$T_{VU}\colon \mathfrak{E}(U) \to \mathfrak{E}(V)$ is completely contractive and

$$T_{VU}(x \cdot a) = T_{VU}(x) \cdot \pi_{VU}(a) \qquad (x \in \mathfrak{E}(U),\ a \in \mathfrak{A}(U)),$$

where $\pi_{VU}\colon \mathfrak{A}(U) \to \mathfrak{A}(V)$ are the restriction maps in $\mathfrak{A}$.

The categories we are interested in

Categories we need to consider
(for a given sheaf $\mathfrak{A}$ of C^*-algebras on X):

$\mathcal{OM}od_{\mathfrak{A}}^{\infty}(X)$	the category with objects the right operator $\mathfrak{A}$-modules and morphisms (at U) the completely bounded $\mathfrak{A}(U)$-module maps;
$\mathcal{OM}od_{\mathfrak{A}}^{1}(X)$	the subcategory of $\mathcal{OM}od_{\mathfrak{A}}^{\infty}(X)$ with the same objects and morphisms (at U) the completely contractive $\mathfrak{A}(U)$-module maps.

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(for a given sheaf $\mathfrak{A}$ of C^*-algebras on X):

$\mathscr{OMod}_{\mathfrak{A}}^{\infty}(X)$	the category with objects the right operator $\mathfrak{A}$-modules and morphisms (at U) the completely bounded $\mathfrak{A}(U)$-module maps; additive, finitely bicomplete
$\mathscr{OMod}_{\mathfrak{A}}^{1}(X)$	the subcategory of $\mathscr{OMod}_{\mathfrak{A}}^{\infty}(X)$ with the same objects and morphisms (at U) the completely contractive $\mathfrak{A}(U)$-module maps. non-additive, bicomplete

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$\mathcal{OM}od_{\mathfrak{A}}^{\infty}(X)$ — the category with objects the right operator $\mathfrak{A}$-modules and morphisms (at U) the completely bounded $\mathfrak{A}(U)$-module maps;

we need this one for homological algebra

$\mathcal{OM}od_{\mathfrak{A}}^{1}(X)$ — the subcategory of $\mathcal{OM}od_{\mathfrak{A}}^{\infty}(X)$ with the same objects and morphisms (at U) the completely contractive $\mathfrak{A}(U)$-module maps.

we need this one for constructions

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$\mathscr{OMod}_{\mathfrak{A}}^{1}(X)$	the subcategory of $\mathscr{OMod}_{\mathfrak{A}}^{\infty}(X)$ with the same objects and morphisms (at U) the completely contractive $\mathfrak{A}(U)$-module maps.

None of these categories is abelian!

The morphisms

Let $\mathfrak{E}, \mathfrak{F} \in \mathscr{O}\mathscr{M}od^{\infty}_{\mathfrak{A}}(X)$. A morphism $\varphi\colon \mathfrak{E} \to \mathfrak{F}$ is a natural transformation between the functors $\mathfrak{E}$ and $\mathfrak{F}$. In $\mathscr{O}\mathscr{M}od^{\infty}_{\mathfrak{A}}(X)$, this means we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{E}(U) & \xrightarrow{\varphi_U} & \mathfrak{F}(U) \\ {\scriptstyle T_{VU}}\downarrow & & \downarrow{\scriptstyle S_{VU}} \\ \mathfrak{E}(V) & \xrightarrow[\varphi_V]{} & \mathfrak{F}(V) \end{array}$$

for each $U \in \mathscr{O}_X$, where each φ_U is a completely bounded $\mathfrak{A}(U)$-module map and $\|\varphi\|_{cb} := \sup_{U \in \mathscr{O}_X} \|\varphi_U\|_{cb} < \infty$. In $\mathscr{O}\mathscr{M}od^{1}_{\mathfrak{A}}(X)$, in addition, $\|\varphi\|_{cb} \leq 1$.

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Notation: $CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$ respectively $CB^1_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$.

The morphisms

Proposition

Let $\mathfrak{E}$ be a presheaf of operator $\mathfrak{A}$-modules on X. For each $t \in X$, the stalk E_t belongs to $\mathscr{OMod}^1_{\mathsf{A}_t}$.

The morphisms

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Proposition

Let $\mathfrak{E}$, $\mathfrak{F}$ be presheaves of operator $\mathfrak{A}$-modules on X and let $\varphi \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$. For each $t \in X$, we have $\varphi_t \in CB_{\mathsf{A}_t}(\mathsf{E}_t, \mathsf{F}_t)$. Moreover, φ is completely contractive if and only if φ_t is completely contractive for all $t \in X$.

The isomorphisms

Theorem

Let $\varphi \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$ for some $\mathfrak{E}, \mathfrak{F} \in \mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$. Then the following conditions are equivalent.

(a) *φ is an isomorphism;*

(b) *φ_U is an isomorphism in $\mathscr{OMod}^{\infty}_{\mathfrak{A}(U)}$ for each $U \in \mathscr{O}_X$ and $\sup_{U \in \mathscr{O}_X} \|\varphi_U^{-1}\|_{cb} < \infty$;*

(c) *φ_t is an isomorphism in $\mathscr{OMod}^{\infty}_{\mathrm{A}_t}$ for each $t \in X$, $\sup_{t \in X} \|\varphi_t^{-1}\|_{cb} < \infty$ and φ_t^0 is surjective for each $t \in X$.*

here, $\varphi_t^0 \colon \mathrm{E}_t^0 \to \mathrm{F}_t^0$ is the restriction of φ_t to the *uncompleted directed colimit*.

to be continued . . .