Local Multipliers and Derivations, Sheaves of C-Algebras and Cohomology*

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Part V: Towards sheaf cohomology

A key property in abelian categories

in an abelian category,

- the morphism set between any pair of objects is an abelian group;

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A key property in abelian categories

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- the morphism set between any pair of objects is an abelian group;
- every morphism has a kernel and a cokernel;
- every morphism can be uniquely factorised as

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & F \\ & G \arrow[ur, "\mu"'] & \end{tikzcd}$$

where π is an epimorphism and μ is a monomorphism.

A replacement for the key property

to make up for the missing third property in $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$, we introduce an exact structure.

Kernels in $\mathcal{OMod}_{\mathfrak{A}}^{\infty}(X)$

Definition

Let $\varphi \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$ for some $\mathfrak{E}, \mathfrak{F} \in \mathcal{OMod}_{\mathfrak{A}}^{\infty}(X)$. For each $U \in \mathcal{O}_X$, we set $(\operatorname{Ker} \varphi)(U) = \ker \varphi_U$.

Since $T_{VU} \ker \varphi_U \subseteq \ker \varphi_V$ whenever $V \subseteq U$, we can restrict the connecting morphisms T_{VU} in $\mathfrak{E}$ to connecting morphisms $\ker \varphi_U \to \ker \varphi_V$. In this way, we obtain a sub-presheaf $\operatorname{Ker} \varphi$ of $\mathfrak{E}$ which is easily checked to be a sheaf.

We will call this the *sheaf kernel* or simply the *kernel* of φ.

Cokernels in $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$

Definition

Let $\varphi \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$ for some $\mathfrak{E}, \mathfrak{F} \in \mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$. For each $U \in \mathscr{O}_X$, we set $(\operatorname{PCoker}\varphi)(U) = \operatorname{coker}\varphi_U$. Recall that $\operatorname{coker}\varphi_U = \mathfrak{F}(U)/\overline{\operatorname{im}\varphi_U}$ by definition.

Since $S_{VU}\operatorname{im}\varphi_U \subseteq \operatorname{im}\varphi_V$ whenever $V \subseteq U$, the connecting morphisms S_{VU} in $\mathfrak{F}$ induce connecting morphisms $\operatorname{coker}\varphi_U \to \operatorname{coker}\varphi_V$. In this way, we obtain a presheaf $\operatorname{PCoker}\varphi$, called the *presheaf cokernel* of φ.

Since this is in general not a sheaf, we define the *sheaf cokernel* or simply the *cokernel* of φ as the sheafification of $\operatorname{PCoker}\varphi$: $\operatorname{Coker}\varphi = (\operatorname{PCoker}\varphi)^{\sim}$.

Kernels and cokernels in $\mathscr{OMod}_{\mathfrak{A}}^{\infty}(X)$

Remarks

- The canonical embedding $\iota\colon \operatorname{Ker}\varphi \to \mathfrak{E}$ is in fact a monomorphism in $\mathscr{OMod}_{\mathfrak{A}}^{1}(X)$ since each ι_U is a complete isometry.
- The canonical quotient morphism $\pi\colon \mathfrak{F} \to \operatorname{PCoker}\varphi$ is in fact an epimorphism in $\mathscr{OMod}_{\mathfrak{A}}^{1}(X)$ since each π_U is a completely contractive complete quotient mapping.

  Letting $\tilde{\pi}\colon \mathfrak{F} \to \operatorname{Coker}\varphi$ be given by $\tilde{\pi}_U = \eta_U \pi_U$, $U \in \mathscr{O}_X$, where η is the sheafification transformation, we have $\|\tilde{\pi}\|_{cb} \leq 1$, since $\|\tilde{\pi}_U\|_{cb} \leq \|\eta_U\|_{cb}\,\|\pi_U\|_{cb}$ for all $U \in \mathscr{O}_X$.

Kernels and cokernels in $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$

Theorem

Let X be a topological space and let $\mathfrak{A}$ be a sheaf of C-algebras on X. Let $\mathfrak{E}, \mathfrak{F} \in \mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$. For each $\varphi \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$, $\operatorname{Ker}\varphi$ is a kernel in the category $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$ and $\operatorname{Coker}\varphi$ is a cokernel in $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$. In particular, the morphisms ι and $\tilde{\pi}$ defined above are a monomorphism and an epimorphism, respectively in $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$.*

A glimpse into the proof

$$
\begin{tikzcd}
\ker\varphi_U \arrow[rr, "\iota_U"] \arrow[ddd, "T'_{VU}"'] & & \mathfrak{E}(U) \arrow[r, "\varphi_U"] \arrow[ddd, "T_{VU}"] & \mathfrak{F}(U) \arrow[ddd, "S_{VU}"] \\
& \mathfrak{G}(U) \arrow[ul, "\rho^{(U)}"] \arrow[ur, "\psi_U"] \arrow[urr, "\varphi_U\psi_U=0"'] \arrow[d, "R_{VU}"'] & & \\
& \mathfrak{G}(V) \arrow[dl, "\rho^{(V)}"'] \arrow[dr, "\psi_V"] \arrow[drr, "\varphi_V\psi_V=0"] & & \\
\ker\varphi_V \arrow[rr, "\iota_V"'] & & \mathfrak{E}(V) \arrow[r, "\varphi_V"'] & \mathfrak{F}(V)
\end{tikzcd}
$$

A kernel

Let $f \in \mathrm{Mor}_{\mathscr{C}}(A, B)$ for some $A, B \in \mathscr{C}$.

A morphism $i\colon K \to A$ is a *kernel* of f if $fi = 0$ and for each $D \in \mathscr{C}$ and $g \in \mathrm{Mor}_{\mathscr{C}}(D, A)$ with $fg = 0$ there is a unique $h \in \mathrm{Mor}_{\mathscr{C}}(D, K)$ making the diagram below commutative

$$\begin{tikzcd}
& D \arrow[dl, dashed, "h"'] \arrow[d, "g"'] \arrow[dr, bend left, "0"] & \\
K \arrow[r, "i"] \arrow[rr, bend right, "0"'] & A \arrow[r, "f"] & B
\end{tikzcd}$$

Any kernel is a monomorphism and is, up to isomorphism, unique.

A cokernel

Let $f \in \mathrm{Mor}_{\mathscr{C}}(A, B)$ for some $A, B \in \mathscr{C}$.

A morphism $p\colon B \to C$ is a *cokernel* of f if $pf = 0$ and for each $D \in \mathscr{C}$ and $g \in \mathrm{Mor}_{\mathscr{C}}(B, D)$ with $gf = 0$ there is a unique $h \in \mathrm{Mor}_{\mathscr{C}}(C, D)$ making the diagram below commutative

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left, "0"] \arrow[dr, bend right, "0"'] & B \arrow[r, "p"'] \arrow[d, "g"'] & C \arrow[dl, dashed, "h"] \\
 & D &
\end{tikzcd}$$

Any cokernel is an epimorphism and is, up to isomorphism, unique.

Kernels and cokernels in $\mathcal{OMod}^{\infty}_{\mathfrak{A}}(X)$

Fact.

Let $\mathscr{C}$ be a category with the property that *every* morphism has both a kernel and a cokernel (which is the case in $\mathcal{OMod}^{\infty}_{\mathfrak{A}}(X)$ and in $\mathcal{OMod}^{1}_{\mathfrak{A}}(X)$).

Then a morphism which is a kernel necessarily is the kernel of its cokernel, and a morphism which is a cokernel necessarily is the cokernel of its kernel.

Exact categories

let $\mathscr{C}$ be an additive category;

Exact categories

let $\mathscr{C}$ be an additive category;

if $\mathscr{C}$ is abelian

every monomorphism is a kernel and every epimorphism is a cokernel;

Exact categories

let $\mathscr{C}$ be an additive category;

in general, a monomorphism which is a kernel is called admissible

$$E \rightarrowtail F$$

and an epimorphism which is a cokernel is called admissible

$$E \twoheadrightarrow F$$

A *kernel–cokernel pair* (M, P) consists of two composable morphisms in $\mathscr{C}$ such that $M = \operatorname{Ker} P$ and $P = \operatorname{Coker} M$,

$$E_1 \overset{M}{\rightarrowtail} E_2 \overset{P}{\twoheadrightarrow} E_3$$

where $E_i \in \mathscr{C}$.

Exact categories

An *exact structure* on an additive category $\mathscr{C}$ is a class of kernel–cokernel pairs, closed under isomorphisms, such that the following axioms are satisfied.

[E0] $\forall\ E \in \mathscr{C}$: 1_E is an admissible monomorphism;

[E0$^{\mathrm{op}}$] $\forall\ E \in \mathscr{C}$: 1_E is an admissible epimorphism;

[E1] the class of admissible monomorphisms is closed under composition;

[E1$^{\mathrm{op}}$] the class of admissible epimorphisms is closed under composition;

[E2] the push-out of an admissible monomorphism along an arbitrary morphism exists and yields an admissible monomorphism;

[E2$^{\mathrm{op}}$] the pull-back of an admissible epimorphism along an arbitrary morphism exists and yields an admissible epimorphism.

An exact structure on $\mathcal{OMod}_{\mathfrak{A}}^{\infty}(X)$

Definition

Let X be a topological space and let $\mathfrak{A}$ be a sheaf of C^*-algebras on X. Let $\mathcal{Ex}_{\mathfrak{A}}(X)$ denote the collection of all kernel–cokernels pairs in $\mathcal{OMod}_{\mathfrak{A}}^{\infty}(X)$

$$\mathfrak{E}_1 \overset{\mu}{\rightarrowtail} \mathfrak{E}_2 \overset{\varpi}{\twoheadrightarrow} \mathfrak{E}_3 \ .$$

We call this the *canonical exact structure* on $\mathcal{OMod}_{\mathfrak{A}}^{\infty}(X)$.

An exact structure on $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$

Definition

Let X be a topological space and let $\mathfrak{A}$ be a sheaf of C^*-algebras on X. Let $\mathscr{Ex}_{\mathfrak{A}}(X)$ denote the collection of all kernel–cokernels pairs in $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$

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We call this the *canonical exact structure* on $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$.

Theorem

The class $\mathscr{Ex}_{\mathfrak{A}}(X)$ of all kernel–cokernel pairs defines an exact structure on $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$.

An exact structure on $\mathscr{O}\mathscr{M}od^{\infty}_{\mathfrak{A}}(X)$

Remarks

1. $\mathscr{E}x_{\mathfrak{A}}(X)$ is the largest exact structure on $\mathscr{O}\mathscr{M}od^{\infty}_{\mathfrak{A}}(X)$.

An exact structure on $\mathscr{O}\mathscr{M}od_{\mathfrak{A}}^{\infty}(X)$

Remarks

1. $\mathscr{E}x_{\mathfrak{A}}(X)$ is the largest exact structure on $\mathscr{O}\mathscr{M}od_{\mathfrak{A}}^{\infty}(X)$.
2. Axioms [E0] and [E0$^{\mathrm{op}}$] are easily verified.

An exact structure on $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$

Remarks

3. Given a kernel–cokernel pair (μ, ϖ) in $\mathscr{Ex}_{\mathfrak{A}}(X)$ we obtain an isomorphic pair $(\iota, \tilde{\pi})$

$$\begin{tikzcd}
\mathfrak{E}_1 \arrow[r, tail, "\mu"] \arrow[d, "\nu", "\cong"'] & \mathfrak{E}_2 \arrow[r, two heads, "\varpi"] \arrow[d, equal] & \mathfrak{E}_3 \arrow[d, "\nu'"', "\cong"] \\
\operatorname{Ker}\pi \arrow[r, tail, "\iota"] & \mathfrak{E}_2 \arrow[r, two heads, "\tilde{\pi}"] & \operatorname{Coker}\iota
\end{tikzcd}$$

For each $U \in \mathscr{O}_X$, we get an exact sequence in $\mathscr{OMod}^{1}_{\mathfrak{A}(U)}$

$$0 \longrightarrow \ker \pi_U \xrightarrow{\iota_U} \mathfrak{E}_2(U) \xrightarrow{\pi_U} \operatorname{coker} \iota_U \longrightarrow 0$$

where $\operatorname{coker} \iota_U = \mathfrak{E}_2(U)/\operatorname{im} \iota_U$ since ι_U is a complete isometry.

Characterising kernels in $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$

Proposition

Let $\mu \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$ for some $\mathfrak{E}, \mathfrak{F} \in \mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$ and let $\tilde{\pi} \colon \mathfrak{F} \to \operatorname{Coker} \mu$. Then μ is a kernel in $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$ if and only if, for each $t \in X$, (μ_t, π_t) is a kernel–cokernel pair in $\mathscr{OMod}^{\infty}_{\mathrm{A}_t}$ with $\operatorname{im} \mu_t^0 = \ker \pi_t^0$ and $\sup_{t \in X} \|\mu_t^{-1}\|_{cb} < \infty$.

$$\begin{tikzcd}
\mathfrak{E} \arrow[rr, "\mu"] \arrow[d, "\psi"'] & & \mathfrak{F} \arrow[r, two heads, "\tilde{\pi}"] & \operatorname{Coker} \mu \\
\operatorname{Ker} \tilde{\pi} \arrow[urr, hook, "\iota"'] & & &
\end{tikzcd}$$

Characterising cokernels in $\mathscr{OM}od^{\infty}_{\mathfrak{A}}(X)$

Proposition

Let $\varpi \in CB_{\mathfrak{A}}(\mathfrak{F}, \mathfrak{G})$ for some $\mathfrak{F}, \mathfrak{G} \in \mathscr{OM}od^{\infty}_{\mathfrak{A}}(X)$. Then ϖ is a cokernel in $\mathscr{OM}od^{\infty}_{\mathfrak{A}}(X)$ if and only if, for each $t \in X$, ϖ^0_t is completely open onto G^0_t and $\sup_{t\in X} \|\varpi_t^{-1}\|_{cb} < \infty$.

$$\begin{tikzcd}
\operatorname{Ker}\varpi \arrow[r, tail, "\iota"] & \mathfrak{F} \arrow[r, "\varpi"] \arrow[dr, "\tilde{\pi}"'] & \mathfrak{G} \\
& & \operatorname{Coker}\iota \arrow[u, "\varphi"']
\end{tikzcd}$$

The isomorphisms in $\mathcal{OM}od^{\infty}_{\mathfrak{A}}(X)$

Theorem

Let $\varphi \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$ for some $\mathfrak{E}, \mathfrak{F} \in \mathcal{OM}od^{\infty}_{\mathfrak{A}}(X)$. Then the following conditions are equivalent.

(a) *φ is an isomorphism;*

(b) *φ_U is an isomorphism in $\mathcal{OM}od^{\infty}_{\mathfrak{A}(U)}$ for each $U \in \mathcal{O}_X$ and $\sup_{U \in \mathcal{O}_X} \|\varphi_U^{-1}\|_{cb} < \infty$;*

(c) *φ_t is an isomorphism in $\mathcal{OM}od^{\infty}_{\mathrm{A}_t}$ for each $t \in X$, $\sup_{t \in X} \|\varphi_t^{-1}\|_{cb} < \infty$ and φ_t^0 is surjective for each $t \in X$.*

here, $\varphi_t^0 \colon \mathrm{E}_t^0 \to \mathrm{F}_t^0$ is the restriction of φ_t to the *uncompleted directed colimit*.

We need to consider the following commutative diagram.

$$\begin{tikzcd}
\mathfrak{E}(U) \arrow[r, "\varphi_U"] \arrow[d, "T_{VU}"] \arrow[dd, bend right, "T_U"'] & \mathfrak{F}(U) \arrow[d, "S_{VU}"] \\
\mathfrak{E}(V) \arrow[r, "\varphi_V"] \arrow[d, "T_V"] & \mathfrak{F}(V) \arrow[d, "S_V"] \\
\mathrm{E}_t^0 \arrow[r, "\varphi_t^0"'] & \mathrm{F}_t^0
\end{tikzcd}$$

where $t \in X$ is fixed and $V, U \in \mathscr{U}_t$, $V \subseteq U$. Moreover, $T_U\colon \mathfrak{E}(U) \to \mathrm{E}_t$ denotes the canonical morphism into the stalk E_t at t and $\mathrm{E}_t^0 = \bigcup_{U \in \mathscr{U}_t} T_U\mathfrak{E}(U)$, the *uncompleted directed colimit*, which is dense in E_t.

Lemma (Axiom [E1])

Let (μ, ϖ) and (μ', ϖ') be two kernel–cokernel pairs in $\mathcal{Ex}_{\mathfrak{A}}(X)$. Suppose that μ and μ' are composable so that we have the commutative diagram

$$\begin{tikzcd}
\mathfrak{F}_3 & & \operatorname{Coker}(\mu\mu') \\
\mathfrak{F}_2 = \mathfrak{E}_1 \arrow[u, two heads, "\varpi'"] \arrow[r, tail, "\mu"] & \mathfrak{E}_2 \arrow[ur, "\varphi"] \arrow[r, two heads, "\varpi"] & \mathfrak{E}_3 \\
\mathfrak{F}_1 \arrow[u, tail, "\mu'"] \arrow[ur, "\mu\mu'"'] & &
\end{tikzcd}$$

Then $\mu\mu'$ is an admissible monomorphism in $\mathcal{OMod}_{\mathfrak{A}}^{\infty}(X)$.

Lemma (Axiom [E1$^{\mathrm{op}}$])

Let (μ, ϖ) and (μ', ϖ') be two kernel–cokernel pairs in $\mathcal{Ex}_{\mathfrak{A}}(X)$. Suppose that ϖ and ϖ' are composable so that we have the commutative diagram

$$\begin{tikzcd}
 & & \mathfrak{F}_1 \arrow[d, tail, "\mu'"] \\
\mathfrak{E}_1 \arrow[r, tail, "\mu"] & \mathfrak{E}_2 \arrow[r, two heads, "\varpi"] \arrow[dr, "\varpi'\varpi"'] & \mathfrak{E}_3 = \mathfrak{F}_2 \arrow[d, two heads, "\varpi'"] \\
 & & \mathfrak{F}_3
\end{tikzcd}$$

Then $\varpi'\varpi$ is an admissible epimorphism in $\mathcal{OMod}^\infty_{\mathfrak{A}}(X)$.

Proposition

Let $\mathfrak{E}$, $\mathfrak{F}$ and $\mathfrak{G} \in \mathscr{OMod}_{\mathfrak{A}}^{\infty}(X)$ and $\tau \in CB_{\mathfrak{A}}(\mathfrak{F}, \mathfrak{G})$, $\rho \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{G})$ be given. Then the pullback of ρ along τ exists and in the pullback diagram

$$\begin{tikzcd} \mathfrak{E} \times_{\mathfrak{G}} \mathfrak{F} \arrow[r, "\overline{\rho}"] \arrow[d, "\overline{\tau}"'] & \mathfrak{F} \arrow[d, "\tau"] \\ \mathfrak{E} \arrow[r, "\rho"'] & \mathfrak{G} \end{tikzcd}$$

we have, for each $U \in \mathscr{O}_X$,

$$(\mathfrak{E} \times_{\mathfrak{G}} \mathfrak{F})(U) = \mathfrak{E}(U) \times_{\mathfrak{G}(U)} \mathfrak{F}(U)$$
$$= \{(x, y) \in \mathfrak{E}(U) \times \mathfrak{F}(U) \mid \rho_U(x) = \tau_U(y)\}$$

and $\overline{\tau}_U(x, y) = x$, $\overline{\rho}_U(x, y) = y$ for all (x, y). Moreover, for each $t \in X$,

$$(\mathfrak{E} \times_{\mathfrak{G}} \mathfrak{F})_t^0 = \mathsf{E}_t^0 \times_{\mathsf{G}_t^0} \mathsf{F}_t^0.$$

Proposition

Let $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G} \in \mathscr{OMod}_{\mathfrak{A}}^{\infty}(X)$ and let $\rho \in CB_{\mathfrak{A}}(\mathfrak{G}, \mathfrak{E})$ and $\sigma \in CB_{\mathfrak{A}}(\mathfrak{G}, \mathfrak{F})$ be given. Then the pushout of σ along ρ exists in $\mathscr{OMod}_{\mathfrak{A}}^{\infty}(X)$:

$$\begin{tikzcd} \mathfrak{G} \arrow[r, "\sigma"] \arrow[d, "\rho"'] & \mathfrak{F} \arrow[d, "\bar{\rho}"] \\ \mathfrak{E} \arrow[r, "\bar{\sigma}"'] & \mathfrak{E} \oplus_{\mathfrak{G}} \mathfrak{F} \end{tikzcd}$$

For $U \in \mathscr{O}_X$,

$$\begin{tikzcd} \mathfrak{G}(U) \arrow[r, "\sigma_U"] \arrow[d, "\rho_U"'] & \mathfrak{F}(U) \arrow[d, "\bar{\rho}_U"] \\ \mathfrak{E}(U) \arrow[r, "\bar{\sigma}_U"'] & \mathfrak{E}(U) \oplus_{\mathfrak{G}(U)} \mathfrak{F}(U) \end{tikzcd}$$

where $\mathfrak{E}(U) \oplus_{\mathfrak{G}(U)} \mathfrak{F}(U) = \big(\mathfrak{E}(U) \oplus \mathfrak{F}(U)\big) \big/ \mathfrak{H}(U)$ *with*

$$\mathfrak{H}(U) = \overline{\big\{\big(\rho_U(z), -\sigma_U(z)\big) \mid z \in \mathfrak{G}(U)\big\}}.$$

Proposition

Let $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G} \in \mathcal{OMod}^{\infty}_{\mathfrak{A}}(X)$ *and let* $\rho \in CB_{\mathfrak{A}}(\mathfrak{G}, \mathfrak{E})$ *and* $\sigma \in CB_{\mathfrak{A}}(\mathfrak{G}, \mathfrak{F})$ *be given. Then the pushout of* σ *along* ρ *exists in* $\mathcal{OMod}^{\infty}_{\mathfrak{A}}(X)$*:*

$$\begin{tikzcd} \mathfrak{G} \arrow[r, "\sigma"] \arrow[d, "\rho"'] & \mathfrak{F} \arrow[d, "\bar\rho"] \\ \mathfrak{E} \arrow[r, "\bar\sigma"'] & \mathfrak{E} \oplus_{\mathfrak{G}} \mathfrak{F} \end{tikzcd}$$

For $t \in X$,

$$\begin{tikzcd} \mathsf{G}_t \arrow[r, "\sigma_t"] \arrow[d, "\rho_t"'] & \mathsf{F}_t \arrow[d, "\bar\rho_t"] \\ \mathsf{E}_t \arrow[r, "\bar\sigma_t"'] & \mathsf{E}_t \oplus_{\mathsf{G}_t} \mathsf{F}_t \end{tikzcd}$$

where $\mathsf{E}_t \oplus_{\mathsf{G}_t} \mathsf{F}_t = \big(\mathsf{E}_t \oplus \mathsf{F}_t\big) \big/ \mathsf{H}_t$ *with*

$$\mathsf{H}_t = \overline{\big\{ \big(\rho_t(z), -\sigma_t(z)\big) \mid z \in \mathsf{H}_t \big\}}$$

and $\bar\sigma_t(x) = (x, 0) + \mathsf{H}_t$, $x \in \mathsf{E}_t$, $\bar\rho_t(y) = (0, y) + \mathsf{H}_t$, $y \in \mathsf{F}_t$.

Lemma (Axiom [E2])

The pushout of an admissible monomorphism in $\mathscr{OMod}_{\mathfrak{A}}^{\infty}(X)$ is an admissible monomorphism.

Lemma (Axiom [E2$^{\mathrm{op}}$])

The pullback of an admissible epimorphism in $\mathscr{OMod}_{\mathfrak{A}}^{\infty}(X)$ is an admissible epimorphism.

$\mathcal{OMod}^{\infty}_{\mathfrak{A}}(X)$ is exact.

Theorem

The class $\mathcal{E}x_{\mathfrak{A}}(X)$ of all kernel–cokernel pairs defines an exact structure on $\mathcal{OMod}^{\infty}_{\mathfrak{A}}(X)$.

Exact sequences

in $\mathscr{E}x_{\mathfrak{A}}(X)$, every kernel–cokernel pair

$$\mathfrak{E}_1 \xrightarrow{\mu} \mathfrak{E}_2 \xrightarrow{\varpi} \mathfrak{E}_3$$

where $\mathfrak{E}_i \in \mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$ is called a short exact sequence.

Exact sequences

in $\mathscr{E}x_{\mathfrak{A}}(X)$, every kernel–cokernel pair

$$\mathfrak{E}_1 \overset{\mu}{\rightarrowtail} \mathfrak{E}_2 \overset{\varpi}{\twoheadrightarrow} \mathfrak{E}_3$$

where $\mathfrak{E}_i \in \mathscr{OM}od_{\mathfrak{A}}^{\infty}(X)$ is called a short exact sequence.
In order to define general exact sequences we first introduce the concept of an admissible morphism.

Exact sequences

in $\mathscr{Ex}_{\mathfrak{A}}(X)$, every kernel–cokernel pair

$$\mathfrak{E}_1 \overset{\mu}{\rightarrowtail} \mathfrak{E}_2 \overset{\varpi}{\twoheadrightarrow} \mathfrak{E}_3$$

where $\mathfrak{E}_i \in \mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$ is called a short exact sequence.

Definition

The morphism $\varphi \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$, $\mathfrak{E}, \mathfrak{F} \in \mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$ is called *admissible* if it can be factorised as

$$\begin{tikzcd} \mathfrak{E} \arrow[rr, "\varphi"] \arrow[dr, two heads, "\varpi"'] & & \mathfrak{F} \\ & \mathfrak{G} \arrow[ur, tail, "\mu"'] & \end{tikzcd}$$

for some admissible monomorphism μ and some admissible epimorphism ϖ in $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$.

Exact sequences

Definition

A sequence of admissible morphisms in $\mathscr{OM}od_{\mathfrak{A}}^{\infty}(X)$

$$\begin{tikzcd} \mathfrak{E}_1 \arrow[rr, "\varphi_1"] \arrow[dr, two heads, "\varpi_1"'] & & \mathfrak{E}_2 \arrow[rr, "\varphi_2"] \arrow[dr, two heads, "\varpi_2"'] & & \mathfrak{E}_3 \\ & \mathfrak{G}_1 \arrow[ur, hook, "\mu_1"'] & & \mathfrak{G}_2 \arrow[ur, hook, "\mu_2"'] & \end{tikzcd}$$

is said to be *exact* if the short sequence $\mathfrak{G}_1 \xrightarrow{\mu_1} \mathfrak{E}_2 \xrightarrow{\varpi_2} \mathfrak{G}_2$ is exact. An arbitrary sequence of admissible morphisms in $\mathscr{OM}od_{\mathfrak{A}}^{\infty}(X)$ is *exact* if the sequences given by any two consecutive morphisms are exact.

Characterising admissible morphisms

Theorem

Let $\varphi \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$ *for some* $\mathfrak{E}, \mathfrak{F} \in \mathscr{OMod}^\infty_{\mathfrak{A}}(X)$*. Then* φ *is admissible if and only if, for each* $t \in X$*,*

- $\varphi_t \in CB_{\mathrm{A}_t}(\mathrm{E}_t, \mathrm{F}_t)$ *is admissible;*
- $\hat{\varphi}^0_t$ *is surjective;*
- $(\operatorname{Ker}\varphi)_t = \ker\varphi_t$, $(\operatorname{Coker}\varphi)_t = \operatorname{coker}\varphi_t$, *and*
- $\sup_{t\in X} \|\varphi_t^{-1}\|_{cb} < \infty$.

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- $\varphi_t \in CB_{\mathsf{A}_t}(\mathsf{E}_t, \mathsf{F}_t)$ *is admissible;*
- $\hat{\varphi}^0_t$ *is surjective;*
- $(\operatorname{Ker}\varphi)_t = \ker \varphi_t$, $(\operatorname{Coker}\varphi)_t = \operatorname{coker}\varphi_t$, *and*
- $\sup_{t \in X} \|\varphi_t^{-1}\|_{cb} < \infty$.

In particular, the stalk functor at $t \in X$ is exact from $\mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$ to $\mathscr{OMod}^{\infty}_{\mathsf{A}_t}$.

Characterising admissible morphisms

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Let $\varphi \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$ for some $\mathfrak{E}, \mathfrak{F} \in \mathcal{OMod}^{\infty}_{\mathfrak{A}}(X)$. Then φ is admissible if and only if, for each $t \in X$,

- $\varphi_t \in CB_{\mathrm{A}_t}(\mathrm{E}_t, \mathrm{F}_t)$ *is admissible;*
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Characterising admissible morphisms

Proposition

For every morphism $\varphi \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$, there exists a unique factorisation $\varphi = \mu\hat{\varphi}\pi$ as given in the commutative diagram below.

$$\begin{tikzcd} \operatorname{Ker}\varphi \arrow[r, "\iota"] & \mathfrak{E} \arrow[rr, "\varphi"] \arrow[d, "\pi"'] & & \mathfrak{F} \arrow[r, "\rho"] & \operatorname{Coker}\varphi \\ & \operatorname{Coker}\iota \arrow[rr, "\hat{\varphi}"] & & \operatorname{Ker}\rho \arrow[u, "\mu"'] & \end{tikzcd}$$

Moreover, φ is admissible if and only if $\hat{\varphi}$ is an isomorphism.

Characterising admissible morphisms

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For every morphism $\varphi \in CB_{\mathfrak{A}}(\mathfrak{E}, \mathfrak{F})$, there exists a unique factorisation $\varphi = \mu\hat{\varphi}\pi$ as given in the commutative diagram below.

$$\begin{tikzcd}
\operatorname{Ker}\varphi \arrow[r, "\iota"] & \mathfrak{E} \arrow[rr, "\varphi"] \arrow[d, "\pi"'] & & \mathfrak{F} \arrow[r, "\rho"] & \operatorname{Coker}\varphi \\
& \operatorname{Coker}\iota \arrow[rr, "\hat{\varphi}"] & & \operatorname{Ker}\rho \arrow[u, "\mu"'] &
\end{tikzcd}$$

Moreover, φ is admissible if and only if $\hat{\varphi}$ is an isomorphism.

this relies on the fact that $\mathcal{OMod}_{\mathfrak{A}}^{\infty}(X)$ is *semi-abelian*;
in an abelian category, $\hat{\varphi}$ is always an isomorphism.

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- to introduce the cohomology groups as the right derived functor of the global section functor applied to an injective resolution of $\mathfrak{F} \in \mathscr{OMod}^{\infty}_{\mathfrak{A}}(X)$

Sheaf Cohomology

$$H^i(X, \mathfrak{F}),\ i \in \mathbb{N}$$

where X is a topological space, $\mathfrak{A}$ a sheaf of C^*-algebras on X and $\mathfrak{F} \in \mathcal{OM}od^{\infty}_{\mathfrak{A}}(X)$.


Thank you!